Poincaré-Sobolev theory without potential theory: connections with Harmonic Analysis

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Abstract This article will explore the connections between ”Harmonic Analysis” and several fundamental estimates, including the ”Poincaré-Sobolev”, ”Trudinger”, and ”John-Nirenberg inequalities”. Our central theme is the ”self-improving property” of generalized Poincaré-type inequalities. Through this exploration, we'll show how these connections eliminate the need to rely on ”Potential Operators”, offering a more flexible approach that yields more precise estimates, especially for singular measures. We'll also outline how certain older results can be modernized, including fractional-type results that improve upon celebrated findings by Bourgain-Brezis-Mironescu.

1 Introduction

The basic theme we will outline is to provide a different perspective for proving classical and non-classical Poincaré-Sobolev type estimates. In this section, we present a brief historical overview that motivates our distinct perspective by showing its intimate connection with Harmonic Analysis

1.1 Motivation: functions with controlled oscillation

We start by recalling that a function f satisfies the **Hölder-Lipschitz** condition in $\mathbb{R}^n$ with exponent α, $0<\alpha\leq 1$, if there exists a constant c such that for any $x,y\in\mathbb{R}^n$

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$$|f(x)-f(y)| \leq c|x-y|^{\alpha}$$

A simple estimate yields that there is a constant c such that the oscillation of f satisfies

$$\frac{1}{|Q|}\int_Q |f(y)-f_Q|\,dy \leq c\,\ell(Q)^{\alpha} \qquad Q \in \mathscr{Q}. \tag{1}$$

$\mathscr{Q}$ denotes the the family of cubes with sides parallel to the axes in $\mathbb{R}^n$. We will be using standard notation like $f_Q := \frac{1}{|Q|}\int_Q f(x)\,dx$, the average of f over the cube Q and $\ell(Q)$ as the sidelength of the cube of Q. Actually, it was shown by S. Campanato around 60 years ago that the (natural) point-wise definition and the averaging type estimate using the concept of oscillation (the left hand side of (1)) are equivalent. This is a very interesting fact that was extended by B. Franchi, G. Lu and R. L. Wheeden in [FLW]. They established the following equivalence between the following two conditions, given $\alpha > 0$, a locally integrable function f, and a nonnegative locally integrable function g:

- 1) There exists a constant C such that for all cubes Q:

$$\frac{1}{|Q|}\int_Q |f-f_Q| \leq c\,\frac{\ell(Q)^{\alpha}}{|Q|}\int_Q g, \tag{2}$$

- 2) There exists a constant C such that for all cubes Q:

$$|f(x)-f_Q| \leq c\,I_{\alpha}(g\chi_Q)(x) \qquad \text{a.e. } x \in Q \tag{3}$$

where I_{α}, $0<\alpha<n$, denotes the fractional integral or Riesz potentials of order α and it is defined by

$$I_{\alpha}f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

Of course, the case $g = |\nabla f|$ above with $\alpha = 1$ is a classical result and it is specially interesting since yields the classical local $(1,1)$ Poincaré inequality: there exists a dimensional constant $c_n > 0$ such that for any Lipschitz function and any cube $Q \subset \mathbb{R}^n$, we have

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \leq c_n\frac{\ell(Q)}{|Q|}\int_Q |\nabla f(x)|dx. \tag{4}$$

The limiting case, $\alpha = 0$ and $g := 1$ makes no sense in (3). However, it is well-defined in (2),

$$\frac{1}{|Q|}\int_Q |f(y)-f_Q|\,dy \leq c,$$

providing a class of functions of different nature denoted by BMO, the Bounded Mean Oscillation space introduced by John–Nirenberg.

1.2 Sub-representation formulas and the Sobolev theorems

As we have mentioned, the subrepresentation formula (3) with $\alpha = 1$ and $g = |\nabla f|$ not only yields the $(1,1)$ Poincaré inequality (4) for cubes, but it also gives us the well-known Poincaré-Sobolev inequalities for $1 \leq p < n$.

$$\left(\frac{1}{|Q|}\int_Q |f - f_Q|^{p^*}\,dx\right)^{1/p^*} \leq c_{n,p}\,\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f|^p\,dx\right)^{1/p} \tag{5}$$

or

$$\left(\int_Q |f - f_Q|^{p^*}\,dx\right)^{\frac{1}{p^*}} \leq c_{n,p}\left(\int_Q |\nabla f|^p\,dx\right)^{\frac{1}{p}}$$

where $p^* = \frac{pn}{n-p}$ is the classical Sobolev exponent, or else

$$\frac{1}{p} - \frac{1}{p^*} = \frac{1}{n}.$$

Notice that these local results yield the classical global results,

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \leq c_{n,p}\,\|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

The typical proof of (5) is based on the operator boundedness,

$$I_1 : L^p(\mathbb{R}^n) \to L^{p^*}(\mathbb{R}^n)$$

which holds in the case $p > 1$. At the endpoint $p = 1$, $1^* = \frac{n}{n-1} = n'$, we cannot use the pointwise estimate since I_1 does not map $L^1(\mathbb{R}^n)$ into $L^{n'}(\mathbb{R}^n)$. I_1 only satisfies the weak endpoint boundedness: $I_1 : L^1(\mathbb{R}^n) \to L^{n',\infty}(\mathbb{R}^n)$. Yet, the Sobolev (strong) inequality still holds in this case by combining this with the "truncation method" of Maz'ja, for which [Ha] provides a very clear exposition. In particular, we have:

$$\|f\|_{L^{n'}(\mathbb{R}^n)} \leq c_n\,\|\nabla f\|_{L^1(\mathbb{R}^n)}. \tag{6}$$

This inequality, however, was derived by Gagliardo using a different method. This estimate, commonly known as the Gagliardo-Nirenberg-Sobolev inequality, was first proved by Gagliardo over 60 years ago using a different method. This inequality is incredibly important; for instance, it's well-known to be equivalent to the Isoperimetric inequality

1.3 The influence of the extrapolation theory

[1]

It is not so well-known that there is an intimate relationship between (6) an the extrapolation method in Harmonic Analysis. This connection was developed in [PR-23]. Indeed, the case $p = 1$ in (5), namely the local version of (6) for cubes, has an extended version for general measures, as proven in [FPW2000],

$$\|f - f_Q\|_{L^{n'}(Q,\mu)} \leq c_n \int_Q |\nabla f(x)| \, (M(\chi_Q \mu)(x))^{\frac{1}{n'}} \, dx, \tag{7}$$

where μ denotes an arbitrary measure in $\mathbb{R}^n$. Observe that the constant is purely dimensional. A variant of this proof can be found in [PR-19]. A key advantage of this estimate is its compatibility with extrapolation ideas from Harmonic Analysis, which reveals that all pertinent information is encapsulated within this measure-dependent estimate. The reference [PR-23] expands in this direction obtaining the following result. The appearance of the factor $(M\mu(x))^{\frac{1}{n'}}$ in (7) gives rise in a natural way to the definition of the $A_{1,n'}$ class of weights. This class is defined as the weights w, such that

$$M(w^{n'}) \leq c w^{n'}$$

Observe that $A_{1,n'}$ is a subclass of A_1. Recall that the a weight w belongs to A_1 if there is a constant c such that $Mw(x) \leq c\, w(x)$ a.e. $x \in \mathbb{R}^n$. Also observe that $A_{1,n'}$ is part of a larger family of weights denoted by A_{p,p^*} and defined by,

$$[w]_{A_{p,p^*}} = \sup_Q \left(\fint_Q w^{p^*} \right) \left(\fint_Q w^{-p'} \right)^{\frac{p^*}{p'}} < \infty. \tag{8}$$

p^* is the usual Sobolev exponent,

$$\frac{1}{p} - \frac{1}{p^*} = \frac{1}{n}.$$

This condition was introduced by B. Muckenhoupt and R. Wheeden in [MW] and was used by E. Harboure, R. Macías, and C. Segovia in [HMS] to derive an off-diagonal extrapolation theorem. (see an updated version of this theorem [LMPT]). In summary, as a consequence of (7) we have the following theorem.

Theorem 1. *Let $p \in [1,n)$ and let $w \in A_{p,p^*}$. Then there exists a constant $c_{n,p}$ such that for any cube Q,*

[1] The content of this section deviates from the central theme of the article; readers who wish to remain focused on the main topic may skip directly to the next section. The point is that sections yields another connection between Harmonic Analysis and the theory of Poincaré-Sobolev inequalities

$$\|w(f-f_Q)\|_{L^{p^*}(Q,dx)} \le c_{n,p}[w]_{A_{p,p^*}}^{\frac{1}{n'}} \left(\int_Q |w\nabla f|^p dx\right)^{\frac{1}{p}}.$$

As a consequence we have the global estimate,

$$\|wf\|_{L^{p^*}(\mathbb{R}^n)} \le c_{n,p}\,[w]_{A_{p,p^*}}^{\frac{1}{n'}}\,\|w\nabla f\|_{L^p(\mathbb{R}^n)}. \tag{9}$$

The primary novelty of this result, distinct from the method per se, is that the constant's exponent $[w]_{A_{p,p^*}}^{\frac{1}{n'}}$ is the sharpest possible.

1.4 Generalized Sobolev theory: avoiding the subrepresentation formulae

It is well-known that estimate (2) satisfies a self-improving property that has attracted the attention of many authors. If f satisfies (2) for all cubes Q, then one can deduce higher L^p–integrability ($p>1$) for f, even though it's initially only locally L^1-integrable. The first result of this type was obtained by L. Saloff-Coste in [SC]. Later, P. Hajlasz and P. Koskela extended it in [HaK] to metric spaces with a doubling measure (X,d,μ). It was later shown in [FPW] that a more general theory exists, allowing the right-hand side of (2) to be of a different nature while still yielding the self-improving phenomenon. In particular, the function's gradient doesn't need to be considered here, as the theory in [FPW] was developed in the context of Spaces of Homogeneous Type. The self-improving property is thus not tied to the gradient's presence on the right-hand side of (4), but rather to a discrete summation condition satisfied by the functional. More precisely, we still be considering inequalities in $\mathbb{R}^n$ of the form:

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \le a(Q) \qquad Q\in\mathscr{Q}, \tag{10}$$

where a is a functional defined over the family of all cubes in $\mathbb{R}^n$. An inequality of this kind, having a local oscillation norm on the left and a functional $a(Q)$ on the right, will be called as a generalized Poincaré inequality.

As an illustration, we can make a variant of (2). Consider a non-negative measure μ in $\mathbb{R}^n$ and a parameter $\alpha>0$, then define the following functional:

$$a_\mu(Q) := \ell(Q)^\alpha\,\frac{\mu(Q)}{|Q|} \tag{11}$$

Now, we fix a locally integrable function f such that

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx \le a_\mu(Q) \qquad Q\in\mathscr{Q} \tag{12}$$

Then, it follows from the results in [FPW] that if f satisfies (12) then the following stronger estimate holds,

$$\|f - f_Q\|_{L^{(\frac{n}{\alpha})',\infty}(Q,\frac{dx}{|Q|})} \leq c_{n,\alpha}\, a_\mu(Q), \qquad Q \in \mathscr{Q}. \tag{13}$$

We are using the following notation for the standard weak weighted normalized norm over a cube $Q \subset \mathbb{R}^n$,

$$|f\|_{L^{r,\infty}(Q,\frac{dx}{|Q|})} := \sup_{\lambda>0} \lambda \left(\frac{|\{x \in Q : |f(x)| > \lambda\}|}{|Q|} \right)^{\frac{1}{r}}.$$

This result with $\alpha = 1$, corresponds to the "weak version" of the classical Poincaré-Sobolev inequality (5) for the case $p = 1$. What makes this result powerful is its validity for a fixed measure μ, eliminating any dependence on f. This removes the requirement to consider a special family of functions involving the gradient in the initial hypothesis (12).

Going a bit further, the reason why a result like (13) holds is that the functional a satisfies the following geometric type condition: there exists a finite positive constant c, such that for any cube $Q \in \mathscr{Q}$ and any family $\{Q_i\}$ of disjoint subcubes of Q

$$\left(\sum_i a(Q_i)^p \frac{|Q_i|}{|Q|} \right)^{\frac{1}{p}} \leq c\, a(Q) \tag{14}$$

with $p = (\frac{n}{\alpha})'$. Actually we can take $c = 1$ in this case. This condition and its variants are central to this theory. This condition, introduced in [FPW] for Spaces of Homogeneous Type, leads to geometrical constants in the result that make it less attractive and applicable than in the case of cubes. Although these geometrical constants were significantly improved in [MP98] (see also [LP]), it remains unclear how to prove this result using sub-representation formulas. However, we remit to [C] for recent improvements using sub-representation formulas in the context of $\mathbb{R}^n$.

A primary question is whether the weak norm $L^{(\frac{n}{\alpha})',\infty}$ can be replaced by the strong norm $L^{(\frac{n}{\alpha})'}$ in (13). The truncation method is not applicable here because the measure is fixed, which prevents the existence of a family of functions satisfying the initial estimate. The question can be rephrased as follows: what additional condition must be imposed on the functional a to directly obtain the strong $L^{(\frac{n}{\alpha})'}$?

As an illustration, we can make a variant of the $(1,1)$ Poincaré inequality (4) combining itself and the definition of the A_1 class of weights. Indeed, splitting the average of $|\nabla f|$ over the cube Q and then using the definition of A_1 we have,

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \leq c_n \frac{\ell(Q)}{|Q|} \int_Q |\nabla f(x)| dx$$

$$= c_n\, \ell(Q) \left(\frac{1}{|Q|} \int_Q |\nabla f(x)|\, dx \right)^{\frac{1}{n}} \left(\frac{1}{|Q|} \int_Q |\nabla f(x)| dx \right)^{\frac{1}{n'}}$$

$$\leq c_n[w]_{A_1}^{\frac{1}{n'}} \ell(Q) \left(\frac{1}{|Q|}\int_Q |\nabla f(x)|\,dx\right)^{\frac{1}{n}} \left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|\,w(x)dx\right)^{\frac{1}{n'}}$$

Define

$$a(Q) = a_f(Q) = c_n\,[w]_{A_1}^{\frac{1}{n'}} \ell(Q) \left(\frac{1}{|Q|}\int_Q |\nabla f(x)|\,dx\right)^{\frac{1}{n}} \left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|\,w(x)dx\right)^{\frac{1}{n'}}$$

Then, as consequence of the main result in [FPW] (see Theorem 2 below, or the sharper version in Theorem 3), we have:

$$\left(\frac{1}{w(Q)}\int_Q |f-f_Q|^{n'}w\,dx\right)^{\frac{1}{n'}} \leq c\,a(Q) \qquad Q\in\mathcal{Q}, \tag{15}$$

where the constant depends on the dimension and the A_1 constant of w. This result holds because the functional a satisfies a weighted version of (14) as defined earlier. That is, for any cube $Q\in\mathcal{Q}$ and any family $\{Q_i\}$ of disjoint subcubes of Q, a finite positive constant c exists such that:

$$\left(\sum_i a(Q_i)^p \frac{w(Q_i)}{w(Q)}\right)^{\frac{1}{p}} \leq c\,a(Q) \tag{16}$$

with $p = n'$. As a consequence we derive the global estimate for C^1 functions with compact support,

$$\|f\|_{L^{n'}(w)} \leq c_{n,[w]_{A_1}} \|\nabla f\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n}} \|\nabla f\|_{L^1(w)}^{\frac{1}{n'}}.$$

In [PR-19], a variant of condition (16) was introduced. While not satisfied by (11) at the endpoint $(\frac{n}{\alpha})'$, it holds for indices below it, yielding more precise results, especially when appropriate classes of weights–like the example we just considered and the one we'll address next–are included.

Let's now consider another relevant family of functionals, derived by combining (4) with the following well-known general fact: for any non-negative function g and any A_p weight w,

$$\frac{1}{|Q|}\int_Q g\,dx \leq [w]_{A_p}^{\frac{1}{p}} \left(\frac{1}{w(Q)}\int_Q g(x)^p\,w(x)dx\right)^{\frac{1}{p}}. \tag{17}$$

Then we have that

$$\fint_Q |f(x)-f_Q|dx \leq c_n[w]_{A_p}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p w\,dx\right)^{1/p} \tag{18}$$

and define

$$a_f(Q) := c_n[w]_{A_p}^{\frac{1}{p}} \ell(Q) \left(\frac{1}{w(Q)} \int_Q |\nabla f(x)|^p w\, dx \right)^{1/p}$$

or more generally we could consider for a fixed measure μ and function f such that

$$\fint_Q |f(x) - f_Q| dx \leq a_\mu(Q)$$

where

$$a_\mu(Q) =: \lambda\, \ell(Q) \left(\frac{\mu(Q)}{w(Q)} \right)^{1/p}$$

for some parameter $\lambda \in (0,\infty)$. As some examples before, μ does not depend on f.

Again, the key to gaining integrability for the function on the left lies in the geometrical condition (16) satisfied by a:

$$\left(\sum_i a_\mu(Q_i)^p \frac{w(Q_i)}{w(Q)} \right)^{\frac{1}{p}} \leq a_\mu(Q).$$

Actually, there is a better integrability result (more in the line of Poincaré-Sobolev) since for some larger exponent $q > p$ the functional a_μ satisfies an stronger condition,

$$\left(\sum_i a_\mu(Q_i)^q \frac{w(Q_i)}{w(Q)} \right)^{\frac{1}{q}} \lesssim a_\mu(Q).$$

For the precise definition, we refer to Definition 1.

2 The general theory and the D_p condition

2.1 The D_p condition and examples

Recall that we are considering here the simplest context, namely $\mathbb{R}^n$ with the usual family of cubes $\mathscr{Q}$ and that we are considering "**abstract**" functionals a, namely

$$a : \mathscr{Q} \to (0,\infty)$$

We want to show that if a locally integrable function f satisfies

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \leq a(Q) \qquad Q \in \mathscr{Q}, \tag{19}$$

then one can deduce higher L^p–integrability of f. As already mentioned, these generalized Poincaré inequalities and self-improving results were first considered in a more general setting in [FPW]. The main results were improved in [MP98]. In these papers, the key condition for the self-improving property is that the functional a satisfies the following geometric discrete summation condition.

Definition 1. Let w be any weight in $\mathbb{R}^n$. We say that the functional a satisfies the weighted $D_p(w)$ condition for $0 < p < \infty$ if there is a constant C such that, for any cube Q and any family $\{Q_j\}_j$ of pairwise disjoint subcubes of Q, the following inequality holds:

$$\left(\sum_j a_\mu(Q_j)^p \frac{w(Q_j)}{w(Q)}\right)^{\frac{1}{p}} \leq c\, a_\mu(Q). \tag{20}$$

The best possible constant c above is denoted by $\|a\|_{D_p(w)}$, and also we will write in this case that $a \in D_p(w)$.

The condition is local in nature, somewhat resembling the Carleson condition. For convenience, we'll denote the best constant C as $\|a\|$. Observe that $\|a\| \geq 1$ and that by Hölder's inequality, the family $\{D_r\}$ is decreasing as r increases, that is, if $r < s$, $D_s \subset D_r$. Hence for a functional $a \in D_r$ we can define the optimal exponent to which it belongs, namely

$$r_a = \sup\{r : a \in D_r\}.$$

This exponent can be considered a generalized Sobolev exponent for the functional a. For instance, inspired by (11), we can define for a measure μ and a positive number α:

$$a_\mu(Q) = \ell(Q)^\alpha \left(\frac{\mu(Q)}{|Q|}\right)^{1/p}.$$

Then it is easy to check that $a_\mu \in D_{p^*_\alpha}$ where p^*_α is the usual Sobolev exponent

$$\frac{1}{p} - \frac{1}{p^*_\alpha} = \frac{\alpha}{n}$$

Indeed, we have to show that $a_\mu \in D_{p^*_\alpha}$, namely for any $Q \in \mathscr{Q}$ and any family of pairwise disjoint subcubes $\{Q_j\}$ of Q

$$\left(\sum_j a_\mu(Q_j)^{p^*_\alpha} \frac{|Q_j|}{|Q|}\right)^{\frac{1}{p^*_\alpha}} \leq c\, a_\mu(Q)$$

Indeed, by the definition of p^*_α we have

$$\sum_j a_\mu(Q_j)^{p^*_\alpha}|Q_j| = \sum_j \left(\int_{Q_j} d\mu(y)\right)^{\frac{p^*_\alpha}{p}} \leq \left(\sum_j \int_{Q_j} d\mu(y)\right)^{\frac{p^*_\alpha}{p}} \leq \left(\int_Q d\mu(y)\right)^{\frac{p^*_\alpha}{p}}$$

$$= a_\mu(Q)^{p^*_\alpha}|Q|$$

It's straightforward to check that $\left\|a_\mu\right\| = 1$ and that p_α^* is the sharpest exponent. This exponent's coincidence with the Sobolev exponent (think of the case $\alpha = 1$) is no mere coincidence.

A further illustration comes from a variant of the one leading to (15). Indeed, define again for $\alpha > 0$

$$a_{\mu_1,\mu_2}(Q) := \lambda\, \ell(Q)^\alpha \left(\frac{\mu_1(Q)}{|Q|}\right)^{\frac{1}{n}} \left(\frac{\mu_2(Q)}{w(Q)}\right)^{\frac{1}{n'}}$$

where μ_1 and μ_2 are two measures and w is a weight w in $\mathbb{R}^n$. Now, we fix a locally integrable function f such that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \le a_{\mu_1,\mu_2}(Q) \qquad Q \in \mathscr{Q}. \tag{21}$$

Then we can show that if $w \in A_1$, then

$$\left(\sum_i a(Q_i)^p \frac{w(Q_i)}{w(Q)}\right)^{\frac{1}{p}} \le c\, a_{\mu_1,\mu_2}(Q)$$

with $p = (\frac{n}{\alpha})'$.

Another family of examples with weights which are important in applications are obtained by combining the A_p, $1 \le p < \infty$ condition and the Poincaré $(1,1)$ estimate,

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \le c_n\, [w]_{A_p}^{\frac{1}{p}}\, \ell(Q) \left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p w\, dx\right)^{1/p}$$

We can consider the more general functional

$$a(Q) = \ell(Q)\left(\frac{\mu(Q)}{w(Q)}\right)^{1/p}$$

It is easy to check that $a \in D_p(w)$ but we have something better: let $w \in A_q$ with $1 \le q \le p$. Define the exponent p_w^* by the formula

$$\frac{1}{p} - \frac{1}{p_w^*} = \frac{1}{nq}.$$

Then $a \in D_{p_w^*}(w)$ and further

$$\|a\|_{D_{p_w^*}(w)} \le [w]_{A_q}^{\frac{1}{nq}}$$

namely for any family $\{Q_j\}$ of pairwise disjoint subcubes of Q:

$$\left(\sum_i a(Q_j)^{p_w^*} \frac{w(Q_j)}{w(Q)}\right)^{\frac{1}{p_w^*}} \le [w]_{A_q}^{\frac{1}{nq}}\, a(Q).$$

We can be found the details of this estimate in [PR-19].

Finally, by adapting the proof of (15) but using (17), we can derive the following for any $w \in A_p$:

$$\left(\frac{1}{w(Q)}\int_Q |f-f_Q|^{pn'} w\,dx\right)^{\frac{1}{pn'}} \leq c\,\ell(Q)\left(\frac{1}{|Q|}\int_Q |\nabla f(x)|\,dx\right)^{\frac{1}{n}}\left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^p w(x)dx\right)^{\frac{1}{pn'}}, \tag{22}$$

where the constant c depends on the A_p constant of the weight w. Observe that in this case the functional a satisfies the $D_{n'p}(w)$ condition and we can apply Theorem 2 below. As a consequence, we derive the global estimate for C^1 functions with compact support and for any A_p weight w,

$$\|f\|_{L^{pn'}(w)} \leq c_{n,[w]_{A_p}} \|\nabla f\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n}} \|\nabla f\|_{L^p(w)}^{\frac{1}{n'}}.$$

2.2 A “rough” result

To state our theorems we need to recall the well known class of weights A_∞, introduced by B. Muckenhoupt. It is commonly defined as follows:

$$A_\infty = \bigcup_{p\geq 1} A_p.$$

We will use the following constant, introduced in [HP]:

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)}\int_Q M(w\chi_Q)(x)dx, \tag{23}$$

which seems to be the optimal one.

The result we state now is the version of the original theorem given in [FPW] but in the context of $\mathbb{R}^n$ with cubes. Nevertheless, the results come out with uncontrolled constants.

Theorem 2. *(first version) Let* $a \in D_p(w)$ *with* $w \in A_\infty$. *Let* f *such that*

$$⨍_Q |f(x)-f_Q|\,dx \leq a(Q) \qquad Q \in \mathscr{Q}$$

Then

$$\|f-f_Q\|_{L^{p,\infty}(Q,\frac{wdx}{w(Q)})} \lesssim a(Q) \qquad Q \in \mathscr{Q} \tag{24}$$

We haven’t been precise about the constant in (24) because there is a much more precise version of this result in Theorem 3.

We’ will ll provide the proof of this result because it illustrates the application of Harmonic Analysis, combining fundamentals of Calderón–Zygmund theory with a suitable variant of the Burkholder-Gundy good-λ inequality. A challenge arises from the lack of precise control over the relevant constant, as stated in Theorem 3. Nevertheless, this approach extends to Spaces of Homogeneous Type.

2.3 A sharper result

A sharper version of Theorem 2 is stated next, providing a more precise control of the relevant constants. It it is taken from [CP].

Theorem 3. *Let* $a \in D_p(w)$ *with* $w \in A_\infty$. *Let* f *such that*

$$\fint_Q |f(x) - f_Q|\, dx \le a(Q) \tag{25}$$

Then

$$\left\| f - f_Q \right\|_{L^{p,\infty}(Q, \frac{wdx}{w(Q)})} \le c_n\, p\, [w]_{A_\infty} \|a\|\, a(Q).$$

As a consequence of Kolmogorov's inequality

$$\left(\frac{1}{w(Q)} \int_Q g(x)^q\, w dx \right)^{1/q} \le (\frac{r}{r-q})^{1/q} \|g\|_{L^{r,\infty}(Q,w)}. \tag{26}$$

we have the following:

$$\left(\frac{1}{w(Q)} \int_Q |f(x) - f_Q|^q w(x) dx \right)^{\frac{1}{q}} \le c_n \left(\frac{p}{p-q} \right)^{\frac{1}{p}} p[w]_{A_\infty} \|a\|_{D_p(w)}\, a(Q),$$

for all $0 < q < p$.

Therefore, the $D_p(w)$ condition implies that (25) self-improves to a weighted L^q estimate for all $0 < q < p$. However, we generally cannot reach the strong $L^p(w)$. This is only possible if Maz'ja's "truncation method" (or a weak-implies-strong argument) applies to the functional $a(Q) := a_f(Q)$ which must depend on f. Once applicable, we can then derive the L^p estimate. For a comprehensive overview of the truncation method, see [Ha].

The following section shows that there is a condition on the functional a, only slightly stronger than D_p, that allows us to derive the self-improving result for the strong $L^p(w)$ norm, and this functional does not need to depend on f.

2.4 A recent variant of the D_p condition

In [PR-19], the key condition D_p (Definition 1) was redefined a few years ago. This redefinition aimed to produce (*strong*) self-improving L^p estimates with much more precise constant control. The resulting new condition, $SD_p^s(w)$, is stronger than $D_p(w)$ yet holds for the most significant functionals (examples across different settings are available in [PR-19, CMPR, HMPV]).

Definition 2. Let w be any weight in $\mathbb{R}^n$, and let $s>0$. We say that the functional a satisfies the weighted $SD_p^s(w)$ condition for $0<p<\infty$ if there is a constant c such that, for any cube Q and any family $\{Q_j\}$ of pairwise disjoint subcubes of Q, the following inequality holds:

$$\left(\sum_j a(Q_j)^p \frac{w(Q_j)}{w(Q)}\right)^{\frac{1}{p}} \leq c\left(\frac{\left|\bigcup_j Q_j\right|}{|Q|}\right)^{\frac{1}{s}} a(Q). \tag{27}$$

The best possible constant C above is denoted by $\|a\|_{SD_p(w)}$, and also we will write in this case that $a \in SD_p^s(w)$.

Notice the extra factor $\left(\frac{|\bigcup_j Q_j|}{|Q|}\right)^{\frac{1}{s}}$ which is bounded by one and hence this is an stronger condition than D_p. However, the model examples of functionals satisfying this condition include the classical ones among others, $a(Q)=\ell(Q)^{\alpha}\left(\frac{\mu(Q)}{w(Q)}\right)^{\frac{1}{p}}$, where μ is a locally finite measure and w is any weight.

The main self-improving result is that if $w \in A_\infty$ and $a \in SD_p^s(w)$, then initial inequality (10) implies the weighted L^p generalized Poincaré inequality. In [PR-19], the authors conjectured that the A_∞ condition could be removed. This conjecture was positively resolved a few years ago by Lerner, Lorist, and Ombrosi in [LLO], who used sparse domination to obtain an interesting pointwise estimate. They also achieved a better dependence on s in the constant. More recently, these ideas have been expanded in [GLP] in various directions, particularly improving results from both [LLO] and [FPW], among others.

The theorem derived by [LLO] is the following.

Theorem 4. *Let $a \in SD_p^s(w)$ for some $s>0$ and p, and let f be a locally integrable function such that*

$$\fint_Q |f(x)-f_Q|\,dx \leq a(Q).$$

Then,

$$\|f-f_Q\|_{L^p(Q,\frac{w dx}{w(Q)})} \leq c_n(s+1)\|a\|_{SD_p^s(w_\mu)}a(Q), \qquad Q \in \mathcal{Q}$$

3 Proof of the rough result in Theorem 2

Fix a cube Q and for each $\lambda>0$, let

$$\Omega_\lambda = \{x \in Q : M(f-f_Q)(x) > \lambda\}$$

$M := M_Q^d$ is the dyadic Hardy–Littlewood maximal function relative to Q, namely

$$M_Q h(x) = \sup_{\substack{R \in \mathscr{D}(Q) \\ x \in R}} \frac{1}{|R|} \int_R |h|.$$

Here $\mathscr{D}(Q)$ denotes the collection of all dyadic descendants of Q.

Then by the Lebesgue differentiation theorem we have

$$\{x \in Q : |f(x) - f_Q| > \lambda\} \subset \Omega_\lambda,$$

except possibly for a set of measure 0, and will follow from

$$\sup_{\lambda>0} \lambda^r \frac{w(\Omega_\lambda)}{w(Q)} \leq C^r \|a\|^r a(Q)^r \tag{28}$$

Since $w \in A_\infty$, there are constants $C, \delta > 0$ such that

$$w(E) \leq c \left(\frac{|E|}{|Q|}\right)^\delta w(Q) \qquad E \subset Q, \quad Q \in \mathscr{Q}$$

The key estimate is the following inequality of good–λ type:

Let f, Q, Ω_λ and w be as above. Then for all λ, ε, N with $\lambda > 0$, $N > \gamma c_n$ and $0 < \varepsilon \leq \|a\|$,

$$w(\Omega_{N\lambda}) \leq \frac{\varepsilon^\delta c}{(N - c_n)^\delta} w(\Omega_\lambda) + \frac{\|a\|^p}{\lambda^p \varepsilon^p} a(Q)^p w(Q) \tag{29}$$

To prove (28) we use a standard good–λ argument.

Let's prove (29). Assume first $\lambda \leq a(Q)$, then in this case the inequality is immediate since $\Omega_{N\lambda} \subset Q$ and $\varepsilon \leq \|a\|$. Assume now that $\lambda > a(Q)$, then,

$$\lambda > \frac{1}{|Q|} \int_Q |f - f_Q|\, dx.$$

We can consider the Calderón–Zygmund decomposition of $(f - f_Q)\chi_Q$ relative to Q for these λ's. This yields a collection of dyadic subcubes of Q, $\{Q_i\}$, maximal with respect to inclusion, satisfying $\Omega_\lambda = \cup_i Q_i$ and

$$\lambda < \frac{1}{|Q_i|} \int_{Q_i} |f - f_Q|\, dx \leq 2^n \lambda \tag{30}$$

for each integer i. Let $N = 2^n + 1$. Since $\Omega_{N\lambda} \subset \Omega_\lambda$, we have

$$w(\Omega_{N\lambda}) = w(\Omega_{N\lambda} \cap \Omega_\lambda) = \sum_i w(\{x \in Q_i : M((f - f_Q)\chi_Q)(x) > N\lambda\}) =$$

$$= \sum_i w(\{x \in Q_i : M((f - f_Q)\chi_{Q_i})(x) > N\lambda\})$$

by the maximality of each of the cubes Q_i.

Now, for $x \in Q_i$,

$$|f(x) - f_Q|$$
$$\leq |f(x) - f_{Q_i}| + |f_{Q_i} - f_Q| \leq |f(x) - f_{Q_i}| + \frac{2^n}{|Q_i'|}\int_{Q_i'} |f - f_Q|\, dy$$
$$\leq |f(x) - f_{Q_i}(x)| + 2^n \lambda$$

by the maximality of the cubes. Hence, since if $N = 2^n + 1$,

$$\{x \in Q_i : M((f - f_Q)\chi_{Q_i})(x) > N\lambda\} \subset \{x \in Q_i : M((f - f_{Q_i})\chi_{Q_i})(x) > \lambda\}$$

Now let

$$E_{Q_i} := \{x \in Q_i : M((f - f_{Q_i})\chi_{Q_i})(x) > \lambda\}$$

which is the key set to understand. Then

$$w(\Omega_{N\lambda}) \leq \sum_i w(E_{Q_i}) \leq \sum_{\substack{i \\ a(Q_i) < \varepsilon\lambda}} w(E_{Q_i}) + \sum_{\substack{i \\ a(Q_i) \geq \varepsilon\lambda}} w(E_{Q_i}).$$

To estimate the first sum we use the A_∞ condition on the weight w combined with the weak type $(1,1)$ property (with constant one) of M

For the second sum we will use the $D_p(w)$ condition:

$$w(\Omega_{N\lambda}) \leq c \sum_{\substack{i \\ a(Q_i) < \varepsilon\lambda}} \left(\frac{|E_{Q_i}|}{|Q_i|}\right)^{\delta} w(Q_i) + \sum_{\substack{i \\ a(Q_i) \geq \varepsilon\lambda}} \left(\frac{a(Q_i)}{\varepsilon\lambda}\right)^{p} w(Q_i)$$

$$\leq c \sum_{\substack{i \\ a(Q_i) < \varepsilon\lambda}} \left(\frac{1}{\lambda}\, \frac{1}{|Q_i|}\int_{Q_i} |f - f_{Q_i}|\, dx\right)^{\delta} w(Q_i) + \frac{1}{(\varepsilon\lambda)^p} \sum_i a(Q_i)^p w(Q_i)$$

$$\leq c \sum_{\substack{i \\ a(Q_i) < \varepsilon\lambda}} \left(\frac{a(Q_i)}{\lambda}\right)^{\delta} w(Q_i) + \frac{\|a\|^p}{(\varepsilon\lambda)^p} a(Q)^p w(Q)$$

$$\leq c\, \varepsilon^{\delta} \sum_i w(Q_i) + \frac{\|a\|^p}{(\varepsilon\lambda)^p} a(Q)^p w(Q)$$

$$\leq c\, \varepsilon^{\delta}\, w(\Omega_\lambda) + \frac{\|a\|^p}{(\varepsilon\lambda)^p} a(Q)^p w(Q).$$

This proves the desired good–λ inequality (29) and also completes the proof of the Theorem. $\square$

4 Some extensions of the John–Nirenberg estimate

In the next section, we'll present the proof of Theorem 3. This proof requires initial estimates that improve the classical John–Nirenberg estimate using an alternative approach. We will primarily follow the arguments outlined in [CP].

4.1 Extending the classical John–Nirenberg estimate

Theorem 5. *Let f be a locally integrable function. Then for any cube Q, for any $1 \le p < \infty$ and $1 < r < \infty$, the following estimate holds*

$$\left(\frac{1}{w_r(Q)}\int_Q \left(\frac{M_Q(f-f_Q)(x)}{M^\sharp f(x)}\right)^p w(x)dx\right)^{\frac{1}{p}} \le c_n\, pr'. \tag{31}$$

Hence, if further $w \in A_\infty$ we have

$$\left(\frac{1}{w(Q)}\int_Q \left(\frac{M_Q(f-f_Q)(x)}{M^\sharp f(x)}\right)^p w(x)dx\right)^{\frac{1}{p}} \le c_n\, p\,[w]_{A_\infty}. \tag{32}$$

As before, M_Q denotes the local dyadic maximal operator over Q.
Recall the usual notation for the BMO space,

$$\|f\|_{\mathrm{BMO}} := \sup_Q \frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx.$$

Corollary 1. *Let f be a function in* BMO. *Then for any cube Q, for any $1 \le p < \infty$*

$$\left(\frac{1}{w(Q)}\int_Q M_Q(f-f_Q)(x)^p\, w(x)dx\right)^{\frac{1}{p}} \le c_n\, p[w]_{A_\infty}\|f\|_{\mathrm{BMO}}, \tag{33}$$

and hence,

$$\left(\frac{1}{w(Q)}\int_Q |f(x)-f_Q|^p\, w(x)dx\right)^{\frac{1}{p}} \le c_n\, p\,[w]_{A_\infty}\|f\|_{\mathrm{BMO}}. \tag{34}$$

Remark 1. We point out that we do not know how to extend this result into the context of spaces of homogeneous type since it is not clear if our method works.

Remark 2. We remark that the corresponding result replacing the L^p norm by the (larger) Lorentz norm $L^{p,q}$ with $1 \le q < p$ cannot be proved even in the simplest situation $w = 1$ and without M.

Remark 3. Since throughout the proof the only cubes that appear are dyadic descendants of Q, we actually obtain the stronger estimate

$$\left(\frac{1}{w_r(Q)}\int_Q\left(\frac{M_Q(f-f_Q)(x)}{M_Q^\sharp f(x)}\right)^p w(x)dx\right)^{\frac{1}{p}} \le c_n\, pr',$$

where $M_Q^\sharp$ is the sharp operator taking the supremum over dyadic descendants of Q. Since $M_Q^\sharp \le M^\sharp$, this last estimate is stronger.

Remark 4. We also remark that the factor p in (31) (or (32)) it is crucial since it yields the exponential type result as follows.

Corollary 2. *Let f be a locally integrable function. Then we have: Improved John–Nirenberg estimate:*

$$\left\|\frac{M_Q(f-f_Q)}{M^\sharp f}\right\|_{\exp L(Q,\frac{wdx}{w(Q)})} \le c_n\,[w]_{A_\infty}$$

meaning that there exist dimensional constants $c_1, c_2 > 0$ such that

$$w\left(\left\{x\in Q : \frac{M_Q(f-f_Q)}{M^\sharp f(x)} > t\right\}\right) \le c_1\, e^{-c_2\, t/[w]_{A_\infty}}\, w(Q) \qquad t>0.$$

For every cube and $\lambda,\gamma>0$ we have the following good-λ type inequality

$$w\big(\{x\in Q : M_Q(f-f_Q) > \lambda, M^\sharp f(x) \le \gamma\lambda\}\big) \le c_1\, e^{\frac{-c_2}{\gamma[w]_{A_\infty}}}\, w(Q)$$

For the proof it is enough to apply next standard lemma.

Lemma 1. *Suppose that (X,μ) is a probability space and f a non-negative function such that for every $1\le p<\infty$ we have the L^p bound*

$$\left(\int_X f(x)^p d\mu\right)^{\frac{1}{p}} \le \gamma p,$$

for some constant γ independent from p. Then $f\in \exp(L)(X,\mu)$, meaning

$$\mu(\{x\in X : f(x)>t\}) \le e^{-\frac{t}{4\gamma}}, \quad t>0.$$

Proof. We compute

$$\int_X\left(\exp\frac{f(x)}{4\gamma}-1\right)d\mu = \sum_{n=1}^\infty \frac{1}{n!}\int_X\left(\frac{f(x)}{4\gamma}\right)^n d\mu \le \sum_{n=1}^\infty\frac{1}{n!}\left(\frac{n}{4}\right)^n \le 1.$$

Therefore,

$$\begin{aligned}\mu(\{x\in X: f(x)>t\}) &= \mu(\{x\in X: \frac{f(x)}{4\gamma}-\frac{t}{4\gamma}-\log 2>\log 2\})\\ &\le \int_X \left(\exp\big(\frac{f(x)}{3\gamma}-\frac{t}{4\gamma}-\log 2\big)-1\right)d\mu\\ &= 2e^{-\frac{t}{4\gamma}}\int_X\left(\exp\frac{f(x)}{4\gamma}-1\right)d\mu. \qquad \square\end{aligned}$$

5 Proof of the sharp theorem

In this section we will provide the proof of Theorem 3: if $w\in A_\infty$ and a is a functional satisfying condition the $D_p(w)$ from Definition 1 anf if f is a locally integrable function such that for every cube Q,

$$\frac{1}{|Q|}\int_Q |f-f_Q|\le a(Q). \tag{35}$$

Then, there exists a dimensional constant c_n such that for every cube Q,

$$\|f-f_Q\|_{L^{p,\infty}\left(Q,\frac{w}{w(Q)}\right)}\le c_n\, p\,[w]_{A_\infty}\,\|a\|\,a(Q).$$

Remark 5. Again, the interesting question is how can we get strong result for the functional

$$a(Q)=\ell(Q)\left(\frac{\mu(Q)}{w(Q)}\right)^{1/p}$$

Using Kolmogorov's inequality we have the following corollary.

Corollary 3. *As a consequence, if m is a number such that* $0<m<p<\infty$ *we have*

$$\left\|f-f_Q\right\|_{L^m(Q,\frac{wdx}{w(Q)})}\le c_n(\frac{p}{p-m})^{1/m}\,p\,[w]_{A_\infty}\,\|a\|\,a(Q)$$

We will use the following two result.

Theorem 6 (Sharp RHI, [HPR]). *Let* $w\in A_\infty$*, and let*

$$r_w=1+\frac{1}{\tau_n[w]_{A_\infty}-1}$$

where τ_n *is a fixed dimenional constant. Then*

$$\Big(\frac{1}{|Q|}\int_Q w^{r_w}\Big)^{\frac{1}{r_w}}\le\frac{2}{|Q|}\int_Q w$$

Recall that we are using the following constant

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q) < \infty$$

already mentioned in (23).

Proof (Proof of the sharper result Theorem 3).

Now we proceed with the proof. Fix a cube Q, we have to prove that

$$t^r \frac{w(\{x \in Q : |f(x) - f_Q| > t\})}{w(Q)} \leq (c\|a\|)^r [w]_{A_\infty}^r a(Q)^r \tag{36}$$

with c independent of Q, t and $[w]_{A_\infty}$.

Now, for each $t > 0$, we let

$$\Omega_t = \{x \in Q : M_Q(f - f_Q)(x) > t\}.$$

Then by the Lebesgue differentiation theorem

$$\{x \in Q : |f(x) - f_Q| > t\} \subset \Omega_t.$$

We will assume that $t > a(Q)$ since otherwise (36) is trivial. Hence

$$t > a(Q) \geq \frac{1}{|Q|} \int_Q |f - f_Q|$$

and we can consider the Calderón–Zygmund covering lemma of $|f - f_Q|$ relative to Q for these values of t. This yields a collection $\{Q_j\}$ of dyadic subcubes of Q, maximal with respect to inclusion, satisfying $\Omega_t = \cup_j Q_j$ and

$$t < \frac{1}{|Q_j|} \int_{Q_j} |f - f_Q| \leq 2^n t$$

for each j. Now let $q > 1$ be a big enough number that will be chosen in a moment. Since $\Omega_{qt} \subset \Omega_t$, we have that

$$\begin{aligned} w(\Omega_{qt}) &= w(\Omega_{qt} \cap \Omega_t) \\ &= \sum_j w(\{x \in Q_j : M(f - f_Q)(x) > qt\}) \\ &= \sum_j w(\{x \in Q_j : M((f - f_Q)\chi_{Q_j})(x) > qt\}) \end{aligned}$$

where the last equation follows by the maximality of each of the cubes Q_j. Indeed, for any of these j's and $x \in Q_j$, we have

$$\begin{aligned} M(|f-f_Q|\chi_Q)(x) &= \max\{\sup_{\substack{P:x\in P\in\mathscr{D}(Q)\\ P\subseteq Q_j}} \fint_P |f-f_Q|, \sup_{\substack{P:x\in P\in\mathscr{D}(Q)\\ P\supset Q_j}} \fint_P |f-f_Q|\} \\ &= \sup_{\substack{P:x\in P\in\mathscr{D}(Q)\\ P\subset Q_j}} \fint_P |f| \\ &= M((f-f_Q)\chi_{Q_j})(x), \end{aligned}$$

since by the maximality of the cubes Q_j when P is dyadic (relative to Q) containing Q_j then

$$\frac{1}{|P|}\int_P |f-f_Q| \le t.$$

On the other hand for arbitrary x,

$$\begin{aligned} |f(x)-f_Q| &\le |f(x)-f_{Q_j}| + |f_Q - f_{Q_j}| \\ &\le |f(x)-f_{Q_j}| + \frac{1}{|Q_j|}\int_{Q_j} |f-f_Q| \\ &\le |f(x)-f_{Q_j}| + 2^n t \end{aligned}$$

and then for $q = 2^n + 1$ we arrive to

$$w(\Omega_{qt}) \le \sum_j w(E_{Q_j}),$$

where

$$\begin{aligned} E_{Q_j} &= \{x \in Q_j : M_Q(f-f_{Q_j})(x) > t\} \\ &= \{x \in Q_j : M_{Q_j}(f-f_{Q_j})(x) > t\}, \end{aligned}$$

by the maximality of the cubes Q_j. Now we will use the good-λ from Corollary 2. We use the version with the dyadic sharp maximal function in Remark 3. Let $\gamma > 0$ to be chosen later. Then

$$E_{Q_j} \subseteq \{M_{Q_j}(f-f_{Q_j}) > t, M_d^\sharp f \le \gamma t\} \cup \{M_d^\sharp f > \gamma t\} = A_j \cup B_j$$

and therefore

$$w(E_{Q_j}) \le w(A_j) + w(B_j).$$

For A_j-s, let $[w]_{A_\infty} > 1$ be the exponent for the Reverse Hölder inequality for $w \in A_\infty$ as in Theorem 6. Then, using Corollary 2, we have

$$\sum_j w(A_j) \le c_1 e^{-\frac{c_2}{[w]_{A_\infty}\gamma}} \sum_j w(Q_j) = c_1 e^{-\frac{c_2}{[w]_{A_\infty}\gamma}} w(\Omega_t).$$

Remember that $c_1, c_2 > 0$ are dimensional constants. On the other hand, for B_j we can argue as follows. We have

$$\bigcup_j B_j \subseteq \{x \in Q : M_d^\sharp f(x) > \gamma t\} = \bigcup_i R_i,$$

where R_i are the maximal dyadic subcubes of Q such that

$$\gamma t < \frac{1}{|R_i|} \int_{R_i} |f - f_{R_i}|.$$

Now, using the starting point (35), we clearly have

$$\gamma t \leq a(R_i).$$

Therefore, using that a satisfies the $D_r(w)$ condition (20), we have

$$\begin{aligned}
\sum_j w(B_j) &\leq w\big(\{x \in Q : M_d^\sharp f(x) > \gamma t\}\big) \\
&= \sum_i w(R_i) \\
&\leq \left(\frac{1}{\gamma t}\right)^r \sum_i w(R_i) a(R_i)^r \\
&\leq \|a\|^r \left(\frac{1}{\gamma t}\right)^r w(Q) a(Q)^r.
\end{aligned}$$

Now, if we put everything together, we get

$$(qt)^r w(\Omega_{qt}) \leq c_1 (tq)^r e^{-\frac{c_2}{\gamma [w]_{A_\infty}}} w(\Omega_t) + \left(q \frac{\|a\|}{\gamma}\right)^r w(Q) a(Q)^r.$$

Since we have qt on the left and t on the right, we define the function

$$\varphi(N) = \sup_{0 < t \leq N} t^r w(\Omega_t).$$

This function is increasing, so we have

$$\varphi(N) \leq \varphi(Nq) \leq c_1 q^r e^{-\frac{c_2}{\gamma [w]_{A_\infty}}} \varphi(N) + \left(q \frac{\|a\|}{\gamma}\right)^r w(Q) a(Q)^r.$$

The parameter γ is free, and we make the choice so that

$$c_1 q^r e^{-\frac{c_2}{[w]_{A_\infty} \gamma}} = \frac{1}{2},$$

which means

$$\gamma = \frac{c_n}{r [w]_{A_\infty}}.$$

This yields the result, since $\|f-f_Q\|_{L^{r,\infty}(Q,w)} \leq \sup_N \varphi(N)$.

6 Application to the fractional situation and the BBM phenomenon

To provide context, consider a "rough" initial fractional Poincaré inequality. We can immediately see that the (L^1) local oscillation on a cube

$$\fint_Q |f(x)-f_Q|\,dx \approx \fint_Q \fint_Q |f(x)-f(y)|\,dydx$$

and for any $\delta \in (0,\infty)$ this quantity is bounded by

$$\leq c_n \ell(Q)^\delta \fint_Q \fint_Q \frac{|f(x)-f(y)|}{|x-y|^\delta}\,dy\,dx \leq c_n \ell(Q)^\delta \fint_Q \int_Q \frac{|f(x)-f(y)|}{|x-y|^{n+\delta}}\,dy\,dx.$$

This is is often called the (local) fractional **Gagliardo** seminorm, but we view it as a functional acting on cubes. In fact, this functional "interpolates" between the oscillation of f and the Sobolev functional,

$$\ell(Q)\fint_Q |\nabla f(x)|\,dx$$

with the blowing up factor $\frac{1}{1-\delta}$ in front, namely

Lemma 2. *Let* $\delta \in (0,1)$*, then*

$$\ell(Q)^\delta \fint_Q \int_Q \frac{|f(x)-f(y)|}{|x-y|^{n+\delta}}\,dy\,dx \leq \frac{c_n}{1-\delta}\ell(Q)\fint_Q |\nabla f(x)|\,dx.$$

The classical $(1,1)$ Poincaré (4) is a consequence of the following estimate which seems to be contained in the work of J. Bourgain, H. Brezis, and P. Mironescu [BBM1, BBM2]: let $\delta \in (0,1)$, then

$$\fint_Q |f(x)-f_Q|\,dx \leq c_n\,(1-\delta)\,\ell(Q)^\delta \fint_Q \int_Q \frac{|f(x)-f(y)|}{|x-y|^{n+\delta}}\,dy\,dx \tag{37}$$

A different and interesting approach was considered by M. Milman combining ideas from interpolation theory and extrapolation theory [JM], [DM].

As a consequence of (37) we derived in [MPW] the following result which serves as an starting point in a similar way as (18).

Corollary 4. *Let* $0<\delta<1$, $1 \leq p < \infty$, $f \in L^1_{loc}(\mathbb{R}^n)$ *and* $w \in A_p$*. Then there exists a dimensional constant c such that*

$$\fint_Q |f(x)-f_Q|\,dx \le c\,[w]_{A_p}^{\frac{1}{p}} \frac{(1-\delta)^{\frac{1}{p}}}{\delta^{1-\frac{1}{p}}}\ell(Q)^\delta \left(\frac{1}{w(Q)}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\,dy\,w(x)\,dx\right)^{\frac{1}{p}}$$

for every cube $Q\subset\mathbb{R}^n$.

We emphasize that the proof isn't as straightforward as that of (18), as deriving the sharp factor $(1-\delta)^{\frac{1}{p}}$ introduces an added difficulty. However, the extra factor $\delta^{\frac{1}{p}-1}$ that appears in front should not be there.

Applying the main Theorem in [PR-19] or [CP] as done in [HMPV] we can deduce

Theorem 7. *Let $0<\delta<1$ and $1\le p<\frac{n}{\delta}$. Let $w\in A_1$, and let p_δ^* be the fractional Sobolev exponent*

$$\frac{1}{p}-\frac{1}{p_\delta^*}=\frac{\delta}{n}.$$

Then, there exists a constant $c_n>0$ such that for every $Q\in\mathcal{Q}$ and for any $u\in L^1_{loc}loc(\mathbb{R}^n)$,

$$\inf_{c\in\mathbb{R}}\left(\frac{1}{w(Q)}\int_Q |u-c|^{p_\delta^*}w\,dx\right)^{\frac{1}{p_\delta^*}} \le c_n\,p_\delta^*\,(1-\delta)^{\frac{1}{p}}\,[w]_{A_1}^{\frac{\delta}{n}+1+\frac{1}{p}}\ell(Q)^\delta\left(\frac{1}{w(Q)}\int_Q\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+p\delta}}\,dy\,w\,dx\right)^{\frac{1}{p}}.$$

The idea is to **self-improve** starting from

$$\fint_Q |f(x)-f_Q|\,dx \le c_n\,(1-\delta)^{\frac{1}{p}}\,\ell(Q)^\delta\left(\frac{1}{|Q|}\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}}$$

for every cube $Q\subset\mathbb{R}^n$.

As before, this functional satisfies the $D_{p_\delta^*}$ condition from which (for $1\le p<\frac{n}{\delta}$) recalling that where p_δ^* is the usual Sobolev exponent

$$\frac{1}{p}-\frac{1}{p_\delta^*}=\frac{\delta}{n}.$$

Then we have,

Theorem 8.

$$\left(\fint_Q |f(x)-f_Q|^{p_\delta^*}\,dx\right)^{\frac{1}{p_\delta^*}} \le c_n\,p_\delta^*\,(1-\delta)^{\frac{1}{p}}\,c_n\ell(Q)^\delta\left(\fint_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\right)^{\frac{1}{p}}$$

Corollary 5. global case

$$\left(\int_{\mathbb{R}^n} |f(x)|^{p_\delta^*}\,dx\right)^{\frac{1}{p_\delta^*}} \le c_n\,p_\delta^*\,(1-\delta)^{\frac{1}{p}}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\right)^{1/p},$$

We also have weighted versions of these results but we remit to [HMPV, MPW] for details.

7 Generalized Trudinger Inequalities

Recall that the Sobolev Embedding Theorem states that functions in $W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ actually lie in $L^{p^*}_{\text{loc}}$ for $p^* = \frac{np}{n-p}$ when $1 \le p < n$. We have already mentioned that a more precise version of this statement is the following local inequality:

$$\left(\frac{1}{|Q|}\int_Q |f-f_Q|^{p^*}\right)^{\frac{1}{p^*}} \le c_{n,p}\,\ell(Q)(\frac{1}{|Q|}\int_Q |\nabla f|^p)^{\frac{1}{p}} \qquad Q \in \mathscr{Q}.$$

As p approaches n the constant blows up indicating that the limiting case, where $p = n$, i.e., $W^{1,n}_{\text{loc}} \subseteq L^\infty_{\text{loc}}$, is false. The correct result in this instance is that $W^{1,n}_{\text{loc}}$ lies locally in the class $\exp L^{n'}$. The corresponding inequality, called Trudinger's Inequality, is

$$\|f-f_Q\|_{\exp L^{n'}(Q)} \le C\,(\int_Q |\nabla f|^n)^{\frac{1}{n}}. \tag{38}$$

It should be mentioned that it was also derived by Yudovich in [Y]. The norm on the left hand side is the Luxemburg (or Orlicz) norm associated to the function $\Phi(t) = \exp t^{n'} - 1$. Trudinger's inequality has proved to be a key result in the study of parabolic and elliptic equations at the critical index.

As above we will consider one locally integrable function such that for every cube Q

$$\frac{1}{|Q|}\int_Q |f(y)-f_Q|\,dy \le a(Q),$$

where $a : \mathscr{Q} \to [0,\infty)$. Considering the geometrical type conditions discussed in previous sections, it is natural to ask if such a condition exists that self-improves the L^1 norm in (19) to an exponential type integrability.

Definition 3. . Let $1 < r < \infty$. We say that the functional a satisfies the T_r condition if there exists a finite constant c such that for any cube $Q \in \mathscr{Q}$ and any family $\{Q_i\}$ of disjoint subcubes of Q

$$\sum_i a(Q_i)^r \le c^r\,a(Q)^r. \tag{39}$$

We will use $\|a\|$ to denote the smallest constant c for which (39) holds, which is always greater than or equal to 1. The main examples include $a(Q) = \left(\int_Q g^r\right)^{\frac{1}{r}}$ where $g \in L^r_{\text{loc}}(\mathbb{R}^n)$ and, more generally,

$$a(Q) = \mu(Q)^{\frac{1}{r}}$$

where μ is a locally finite measure. Observe that these conditions are increasing, in the sense that $r < s \Rightarrow T_r \subset T_s$. Also observe that this condition is much stronger than any D_p condition since in particular a is increasing.

Theorem 9. *Assume that the functional a satisfies the T_r condition for some $1 < r < \infty$. Let f is a locally integrable function such that for all cubes Q in $\mathbb{R}^n$*

$$\frac{1}{|Q|}\int_Q |f(y) - f_Q|\,dy \le a(Q) \qquad Q \in \mathscr{Q}. \tag{40}$$

Then there exists a constant c independent of f such that

$$\|f - f_Q\|_{\exp L^{r'}(Q,w)} \le C\,a(Q) \qquad Q \in \mathscr{Q}. \tag{41}$$

Note the important fact that the functional a need not depend on f, in contrast to the classical case where the functional is given by $a(Q) = \left(\int_Q |\nabla f|^n\right)^{\frac{1}{n}}$. This theorem, which can be found in [MP02], is actually derived in a much broader context where no differential structure needs to be considered.

A point of interest here is that the limiting "$r = \infty$" case of this theorem directly corresponds to the generalized John–Nirenberg theorem (Corollary 2). Indeed, let T_∞ denote the functionals a such that for some finite constant κ and for arbitrary cubes Q, P with $P \subset Q$, we have $a(P) \le \kappa\, a(Q)$ (actually it is a corollary of Theorem 3 since the functional a satisfies trivially the D_p condition with constant κ). Observe that $T_r \subset T_\infty$ and that in fact the condition T_∞ is simply the limit as $r \to \infty$ of the condition T_r. Recall that the functional $a(Q) \equiv 1$ is the functional associated to the space BMO, satisfies T_∞.

We finish the section by giving an application. Consider the weighted $(1,1)$ Poincaré inequality

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \le c_n\,[w]_{A_1}\,\frac{\ell(Q)}{w(Q)}\int_Q |\nabla f(x)|w\,dx$$

then by Hölder's inequality we have

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \le c_n\,[w]_{A_1}\,\ell(Q)\left(\frac{1}{w(Q)}\int_Q |\nabla f(x)|^n\,w(x)dx\right)^{\frac{1}{n}}.$$

Now, let $w \in A_1$ be non-degenerate, meaning that for some parameter $\varepsilon > 0$ we have the lower bound $w(x) \ge \varepsilon$, *a.e.* in $\mathbb{R}^n$ (this is always possible by simply taking $w + \varepsilon$). Then we have,

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \le c_n\,\frac{[w]_{A_1}}{\varepsilon}\left(\int_Q |\nabla f(x)|^n\,w(x)dx\right)^{\frac{1}{n}}$$

The functional on the right-hand side then trivially satisfies (39) with $r = n$ and a constant less than $c_n\,\frac{[w]_{A_1}}{\varepsilon}$, from which we derive:

$$\|f - f_Q\|_{\exp L^{n'}(Q,w)} \le C\, c_n \,\frac{[w]_{A_1}}{\varepsilon}\, \left(\int_Q |\nabla f(x)|^n\, w(x) dx\right)^{\frac{1}{n}} \quad Q \in \mathscr{Q},$$

as a consequence of Theorem 9.

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References

[BBM1] Bourgain J., Brezis H., and Mironescu P., *Another look at Sobolev spaces*, Optimal control and partial differential equations, pp. 439–455.

[BBM2] Bourgain J., Brezis H., and Mironescu P., *Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications*, vol. 87, 2002, Dedicated to the memory of Thomas H. Wolff, pp. 77–101.

[CP] Canto J. and Perez C., *Extensions of the John–Nirenberg theorem and applications*, Proc. Amer. Math. Soc. 149 (2021), 1507-1525.

[CMPR] Cejas, M. E., Mosquera C., Pérez, C. & Rela, E. *Self-improving Poincaré–Sobolev type functionals in product spaces*, *J. Anal. Math.* (2022).

[C] Claros A., *Degenerate Poincaré-Sobolev inequalities via fractional integration*, J. Funct. Anal. **289** (2025), 111000. https://doi.org/10.1016/j.jfa.2025.111000

[DM] Domínguez O. and Milman M., Personal communication.

[FLW] Franchi B., Lu G. and Wheeden R. L., *A relationship between Poincaré type inequalities and representation formulas in spaces of homogeneous type*, Intern. Math. Res. Not., 1996 **1**, 1–14

[GLP] I. Gardeazabal, E. Lorist, and Pérez C., *Self-improving properties of generalized Poincaré inequalities*, in preparation.

[FPW] Franchi B., Pérez C., and R. L. Wheeden, *Self-improving properties of John-Nirenberg and Poincaré inequalities on spaces of homogeneous type*, J. Funct. Anal. **153** (1998), no. 1, 108–146.

[FPW2000] Franchi B., Pérez C. and Wheeden R. L., *Sharp geometric Poincaré inequalities for vector fields and non–doubling meaures*, Proc. London Math. Soc. (3) **80** (2000), 665–689.

[Ha] Hajlasz P., *Sobolev inequalities, truncation method, and John domains,* Papers on analysis, V. 83, 109-126, Rep. Univ. Jyväskylä Dep. Math. Stat., 83, Univ. Jyväskylä, Jyväskylä, 2001.

[HaK] Hajłasz P. and Koskela P., *Sobolev meets Poincaré*, C. R. Acad. Sci. Paris, **320** (1995), 1211–1215.

[HMS] Harboure E., Macías R., and Segovia C., *Extrapolation results for classes of weights*, Amer. J. Math., **110** (1988), 383-397.

[HMPV] Hurri-Syrjänen R., Martínez J., Pérez C. and Vähäkangas A., *On the BBM-phenomenon in fractional Poincaré-Sobolev inequalities with weights*, Internat. Math. Res. Notices, 2022; https://doi.org/10.1093/imrn/rnac246.

[HP] Hytönen T. P. and Pérez C., Sharp weighted bounds involving A_∞, Anal. PDE, 6 (2013), 777–818.

[HPR] Hytönen T. P. , Pérez C. and Rela E., *Sharp Reverse Hölder property for A_∞ weights on spaces of homogeneous type,* Journal of Functional Analysis **263**, (2012) 3883–3899.

[JM] Jawerth B. and Milman M., *Extrapolation theory with applications*, Mem. Amer. Math. Soc. **89** (1991), no. 440, iv+82.

[LMPT] Lacey M.T., Moen K., Pérez C., and Torres R.H., *Sharp weighted bounds for fractional integral operators.* J. Funct. Anal. 259 (2010), no. 5, 1073–1097.

[LLO] Lerner A. L., Lorist E., and Ombrosi S., *Operator-free sparse domination*, Forum Math. Sigma **10** (2022), Paper No. e15, 28.

[LP] Lerner A. L., and Pérez C. *Self-improving properties of generalized Poincaré type inequalities through rearrangements,* Mathematica Scandinavica, **97**, (2005), 217-234.

[MP98] MacManus P. and Pérez C., *Generalized Poincaré inequalities: sharp self-improving properties*, Internat. Math. Res. Notices (1998), no. 2, 101–116.

[MP02] MacManus P. and Pérez C., *Trudinger Inequalities without Derivatives*, Trans. Amer. Math. Soc. **354** (2002), 1997-2012.

[M] Milman M., *Notes on limits of Sobolev spaces and the continuity of interpolation scales,* Trans. Amer. Math. Soc. **357** (2005), 3425-3442.

[MPW] Myyryläinen K., Pérez C. and Weigt J., *Weighted fractional Poincaré inequalities via isoperimetric inequalities*, Calc. Var. 63, 205 (2024).

[PR-19] Pérez C. and Rela R., *Degenerate poincaré-sobolev inequalities*, Trans. Amer. Math. Soc. DOI: https://doi.org/10.1090/tran/7775 Published electronically: August 5, 2019.

[MW] Muckenhoupt B. and Wheeden R., *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc. 192 (1974) 261-274.

[PR-23] Pérez C. and Rela E. *A tribute to Pola Harboure: Isoperimetric inequalities and the HMS extrapolation theorem,* Rev. Un. Mat. Argentina, **66**, no. 1 (2023), 265-280. (special volume dedicated to Prof. E. Harboure.)

[SC] Saloff-Coste L., *A Note on Poincaré, Sobolev, and Harnack Inequalities*, Inter. Mat. Res. Notices **2** (1992), 27–38

[Y] Yudovich V. I., Dokl. Akad. Nauk SSRR **138** (1961), 805–808.