# Mortal solutions to the Kähler–Ricci flow

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**Abstract** We aim to provide the working Riemannian geometer with an accessible reference to the algebro-geometric and complex-analytic aspects of the Kähler–Ricci flow.

## 1 Basic theory

The more foundational material can be found in [3, 4, 16, 22, 23, 24, 30, 27, 39, 47, 51, 61, 73, 74, 59].

**Housekeeping 1.1.** *We always assume that manifolds are connected, Hausdorff, and paracompact. A compact manifold is always understood to mean closed without boundary. Smooth is understood to be $\mathcal{C}^\infty$-smooth.*

### *1.1 Complex manifolds*

A *complex manifold X* is a smooth manifold with an atlas of charts $\varphi_\alpha : \mathcal{U}_\alpha \to \mathbf{B}^n \subset \mathbf{C}^n$ such that the transition maps $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \longrightarrow \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ are holomorphic. If $(z^1, ..., z^n)$ are coordinates on $\mathbf{C}^n$, a smooth function $f : \Omega \subseteq \mathbf{C}^n \to \mathbf{C}^n$ on some domain $\Omega$ given by $f(z) = (f^1(z), ..., f^n(z))$ is *holomorphic* if the Cauchy–Riemann equations hold in a neighborhood of every point of $\Omega$, i.e.,

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$$\bar{\partial} f \;:=\; \sum_{\nu=1}^{n} \frac{\partial f}{\partial \bar{z}^{\nu}} \mathrm{d}\bar{z}^{\nu} \;=\; 0.$$

Further details on holomorphy of functions of several variables can be found in [51].

*Example 1.* Euclidean space $\mathbf{C}^n$ and complex projective space $\mathbf{CP}^n$ are complex manifolds [23]. The complex manifolds that holomorphically embed into $\mathbf{C}^n$ (respectively, $\mathbf{CP}^n$) are called *Stein manifolds* (respectively, *projective manifolds*). We invite the reader to consult [22, 24, 51] for a detailed treatment of Stein manifolds.

**Cautionary Remark 1.2.** *The dimension of a complex manifold refers to the complex dimension, i.e., half the (real) dimension of the underlying smooth manifold. A complex manifold of dimension* 1 *is referred to either as a Riemann surface or a curve*[1]*. A complex manifold of dimension* 2 *is called a complex surface.*

The existence of a holomorphic atlas is equivalent to the existence of a smooth[2] endomorphism on the tangent bundle satisfying an integrability criterion [48].

**Definition 1.** Let $M$ be a smooth manifold. An *almost complex structure* is a smooth map $\mathrm{J} : TM \to TM$ such that $\mathrm{J}^2 = -\mathrm{Id}$. A smooth manifold endowed with an almost complex structure is called an *almost complex manifold*. We say that J is a *complex structure* if

$$[\mathrm{J}u, \mathrm{J}v] \;=\; [u,v] + \mathrm{J}([\mathrm{J}u, v] + [u, \mathrm{J}v])$$

for all $u, v \in TM$.

**Definition 2.** A Riemannian metric on a complex manifold $X := (M, \mathrm{J})$ is said to be *Hermitian* if $\mathrm{g}(\mathrm{J}\cdot, \mathrm{J}\cdot) \;=\; \mathrm{g}(\cdot,\cdot)$. A complex manifold endowed with a Hermitian metric is called a *Hermitian manifold*.

**Cautionary Remark 1.3.** *Definition 2 does not imply that g defines a family of Hermitian forms on each tangent space. If $H : T_x^{\mathbf{C}}X \times T_x^{\mathbf{C}}X \to \mathbf{C}$ is a Hermitian form on the complexified tangent space $T_x^{\mathbf{C}}X := T_xX \otimes_{\mathbf{R}} \mathbf{C}$, then $H(u,v) = g(u,v) - \sqrt{-1}g(Ju,v)$. Thus, a Riemannian metric g is Hermitian if it is the real part of a Hermitian inner product.*

If $(z^1, \ldots, z^n)$ are local coordinates on $X$, a Hermitian metric is locally written

$$\mathrm{g} = \mathrm{g}_{k\bar{\ell}} \mathrm{d}z^k \otimes \mathrm{d}\bar{z}^{\ell},$$

where $\mathrm{g}_{k\bar{\ell}} := \mathrm{g}\left(\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^{\ell}}\right)$. The Hermitian condition forces the vanishing of $\mathrm{g}_{k\ell} = \mathrm{g}\left(\frac{\partial}{\partial z^k}, \frac{\partial}{\partial z^{\ell}}\right) = 0$ and $\mathrm{g}_{\bar{k}\bar{\ell}} = \mathrm{g}\left(\frac{\partial}{\partial \bar{z}^k}, \frac{\partial}{\partial \bar{z}^{\ell}}\right) = 0$.

[1] Some authors reserve the term curve for compact complex manifolds of dimension one, together with an embedding into some complex projective space.

[2] If the endomorphism is real-analytic, e.g., in the case of complex Lie groups, this follows from the classical Frobenius theorem (see, e.g., [7]).

*Remark 1.* Hermitian metrics always exist on a complex manifold: If g is a Riemannian metric, then $\hat{g} := g + J^* g$ is Hermitian.

**Cautionary Remark 1.4.** *If $g$ is a Hermitian metric (with respect to some complex structure $J$), then $\omega(\cdot,\cdot) := g(J\cdot,\cdot)$ is a non-degenerate $(1,1)$-form. In local coordinates,*

$$\omega = \sqrt{-1} g_{k\bar{\ell}} dz^k \wedge d\bar{z}^{\ell}.$$

*It is common practice to refer to this $(1,1)$-form as a metric. Further, $\omega$ is sometimes called the 'Kähler form' even if $g$ is not a Kähler metric.*

**Definition 3.** Let $(X, g)$ be a Hermitian manifold. We say that the metric g is *Kähler* if the associated $(1,1)$-form $\omega$ is closed $d\omega = 0$. A complex manifold that admits a Kähler metric is called a *Kähler manifold.*

**Cautionary Remark 1.5.** *There is an ambiguity in what is meant by the 'volume form' of a Kähler metric. Oftentimes this refers to the (symplectic) volume $\omega^n$, which is related to the Riemannian volume form $dv_g$ by*

$$\omega^n \;=\; n!\; dv_g.$$

The complex structure J splits the *complexified tangent bundle*

$$T^{\mathbf{C}}X \;:=\; TX \otimes_{\mathbf{R}} \mathbf{C} \;\simeq\; T^{1,0}X \oplus T^{0,1}X,$$

where $T^{1,0}X := \{u - \sqrt{-1}Ju : u \in T^{\mathbf{C}}X\}$ and $T^{0,1}X := \{u + \sqrt{-1}Ju : u \in T^{\mathbf{C}}X\}$. This induces a splitting of the complex-valued 1-forms $\Lambda^1_{\mathbf{C}}(X) \simeq \Lambda^{1,0}(X) \oplus \Lambda^{0,1}(X)$.

Further, for the complex-valued $k$-forms, we have $\Lambda^k_{\mathbf{C}}(X) \simeq \bigoplus_{p+q=k} \Lambda^{p,q}(X)$. The smooth sections of $\Lambda^{p,q}(X)$ are called $(p,q)$-*forms*. If $(z^1, \ldots, z^n)$ are local holomorphic coordinates on $X$, a smooth $(p,q)$-form is given by

$$\alpha \;=\; \sum f_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} \mathrm{d}z^{i_1} \wedge \cdots \wedge \mathrm{d}z^{i_p} \wedge \mathrm{d}\bar{z}^{j_1} \wedge \cdots \wedge \mathrm{d}\bar{z}^{j_q}.$$

In the important case $p = q = 1$, a $(1,1)$-form is locally given by

$$\alpha \;=\; \sqrt{-1} \sum_{i,j=1}^{n} \alpha_{i\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j.$$

A $(1,1)$-form $\alpha$ is said to be *real* if $\alpha = \bar{\alpha}$ and *positive* if the matrix $(\alpha_{i\bar{j}})$ is positive-definite.

**Definition 4.** A rank one holomorphic vector bundle $L \to X$ on a complex manifold $X$ is called a *line bundle*. The trivial line bundle is denoted by $\mathcal{O}_X$.

*Remark 2.* The notation $\mathcal{O}_X$ is typically used to describe the sheaf of holomorphic functions on $X$. There is a correspondence between line bundles and rank one locally free sheaves. If $L \to X$ is a line bundle, then a locally free sheaf is given by considering the sheaf of sections of $L$. This explains why $\mathcal{O}_X$ is used to denote the trivial bundle.

**Definition 5.** Let $X$ be a complex manifold with $n = \dim_{\mathbf{C}} X$. The *canonical bundle* $K_X$ is the line bundle $\Lambda_X^{n,0}$ of $(n,0)$-forms. The local sections of $K_X$ are given by

$$f(z^1,...,z^n)\mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n,$$

where $f$ is some locally defined smooth function.

**Definition 6.** The *anti-canonical bundle* $-K_X$ is the line bundle dual to $K_X$, i.e., the line bundle satisfying $-K_X \otimes K_X \simeq \mathcal{O}_X$, where $\mathcal{O}_X$ denotes the trivial line bundle.

*Example 2.* If $X$ is a Riemann surface, i.e., $\dim_{\mathbf{C}} X = 1$, then $K_X$ is the cotangent bundle $\Lambda^{1,0}(X)$ and $-K_X$ is the tangent bundle $T^{1,0}X$.

**Cautionary Remark 1.6.** *There are two notations for tensor product of line bundles: Additive notation $mL := L \otimes \cdots \otimes L$ and multiplicative notation $L^m$ or $L^{\otimes m}$. The (equivalence classes of) line bundles on $X$ form a multiplicative group with respect to $\otimes$ called the Picard group $Pic(X)$ (see, e.g., [23]). The additive notation comes from the equivalence between line bundles and (Weil) divisors, i.e., formal $\mathbf{Z}$-linear combinations of hypersurfaces; the Weil divisors on $X$ form an additive group (see, e.g., [23]).*

There is also a splitting of the exterior derivative $\mathrm{d} = \partial + \bar{\partial}$, where $\partial : \Lambda^{p,q}(X) \to \Lambda^{p+1,q}(X)$ and $\bar{\partial} : \Lambda^{p,q}(X) \to \Lambda^{p,q+1}(X)$. Since $\mathrm{d}^2 = 0$, it follows that $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$.

**Definition 7.** Let $X$ be a compact complex manifold. The *Dolbeault cohomology groups* are defined by

$$H^{p,q}(X) \;:=\; \ker(\bar{\partial})/\mathrm{im}(\bar{\partial}).$$

The integers $h^{p,q}(X) := \dim_{\mathbf{C}} H^{p,q}(X)$ are called the *Hodge numbers*. We also set $H^{1,1}(X,\mathbf{R})$ to be the subspace of $H^{1,1}(X)$ whose cohomology classes are represented by real $(1,1)$-forms, i.e., $\alpha \in \Omega^{1,1}(X)$ such that $\alpha = \overline{\alpha}$.

**Definition 8.** The *Kähler cone* $\mathscr{K}$ of a Kähler manifold $X$ is

$$\mathscr{K} := \left\{ \{\omega\} \in H^{1,1}(X,\mathbf{R}) : \text{for some Kähler metric } \omega \right\}.$$

By Hodge theory, if $X$ is compact, $H^{1,1}(X,\mathbf{R})$ is finite-dimensional. Therefore, $\mathscr{K}$ is an open convex cone in a complex vector space of dimension $h^{1,1}(X)$.

*Remark 3.* In the mirror symmetry literature, $\mathscr{K}$ is described as the 'Kähler moduli space', and $h^{1,1}$ is described as the dimension of the Kähler moduli space.

**Definition 9.** Let $X$ be a complex manifold. A subset $A \subseteq X$ is an analytic subset of $X$ if $A$ is closed and if every point $x_0 \in A$ admits an open neighborhood $U$ and holomorphic functions $f_1,\ldots,f_n \in \mathcal{O}(U)$ such that $A \cap U = \{x \in U : f_1(x) = \cdots = f_n(x) = 0\}$.

**Definition 10.** A meromorphic map is a map $f : X \dashrightarrow Y$ that is defined and holomorphic on the complement of a nowhere dense analytic subset $A \subset X$ such that the closure of the graph of $f$ in $X \times Y$ is an analytic subset. If $f : X \dashrightarrow Y$ is a meromorphic map with meromorphic inverse $f^{-1} : Y \dashrightarrow X$, then $f$ is said to be *bimeromorphic*.

**Cautionary Remark 1.7.** *The notation $\dashrightarrow$ indicates that the map is not a priori globally defined everywhere. In the more algebro-geometric literature, this distinction is also sometimes further emphasized by using the term 'map' in place of 'morphism'.*

*Example 3.* The blow-up (or blow-down) of a point—or more generally, a subvariety[3]. If $X$ is a complex manifold, $\dim_{\mathbf{C}} X = n$, and $x \in X$ is a point, the *blow-up* of $X$ at $x$ is a holomorphic map $\pi : \mathrm{Bl}_x(X) \to X$ from a complex manifold $\mathrm{Bl}_x(X)$ diffeomorphic to $X \sharp \overline{\mathbf{CP}^n}$ such that the *exceptional divisor* $\mathrm{Ex}(\pi) := \pi^{-1}(x) \simeq \mathbf{CP}^{n-1}$, and $\pi|_{X \setminus \mathrm{Ex}(\pi)}$ is a biholomorphism. The normal bundle of $\mathrm{Ex}(\pi) \simeq \mathbf{CP}^{n-1}$ is $\mathcal{O}_{\mathbf{CP}^{n-1}}(-1)$.

Consider the specific case of compact Kähler surfaces. Blowing up a point of a Kähler surface yields a curve $C \simeq \mathbf{CP}^1$ with normal bundle $\mathcal{O}_{\mathbf{CP}^1}(-1)$ as the exceptional divisor. Such curves are called $(-1)$-curves. The following theorem of Enriques shows that the converse is true, namely, that $(-1)$-curves can be *blown down* (see, e.g., [4]).

**Theorem 1.8.** *Let $X$ be a compact Kähler surface. An (irreducible) curve $C \subset X$ can be contracted, i.e., there exists a bimeromorphic map $f : X \to Y$ to a Kähler surface $Y$ such that $f(C)$ is a point, if and only if $C$ is a $(-1)$-curve.*

*Remark 4.* The weak factorization theorem [1] tells us that all bimeromorphic maps between projective manifolds are compositions of (a finite number of) blow-ups and blow-downs.

## *1.2 Curvature of Kähler metrics*

For an arbitrary Hermitian metric g, the Levi-Civita connection, a priori, has no relation to the Hermitian structure.

**Definition 11.** The *Chern connection* of a Hermitian metric g is the unique complex-linear connection $\nabla$ such that $\nabla \mathrm{g} = \nabla \mathrm{J} = 0$ and whose torsion T has vanishing $(1,1)$-part, i.e.,

$$\mathrm{T}(\cdot,\cdot) + \mathrm{T}(\mathrm{J}\cdot, \mathrm{J}\cdot) \;=\; 0.$$

[3] This is sometimes called the center of the blow-up. For example, a blow-up with smooth center means the subvariety that is blown up is a smooth subvariety.

*Remark 5.* The Chern connection can equivalently be described as the unique complex-linear connection $\nabla$ such that $\nabla \mathrm{g} = \nabla \mathrm{J} = 0$ and $\nabla^{0,1} = \bar{\partial}$, where $\bar{\partial}$ denotes the holomorphic structure on the tangent bundle. We invite the reader to consult [20, 12] for more details.

One of the many advantages of considering Kähler metrics, is that they afford the following characterization.

**Theorem 1.9.** *Let D and $\nabla$ denote the Levi-Civita and Chern connection of a Hermitian metric g. Then g is Kähler if and only if $\nabla = D$, i.e., if the torsion of the Chern connection vanishes.*

Theorem 1.9 alleviates a significant number of technicalities that one is faced with for a general Hermitian metric (see, e.g., [40, 12]). In particular, the curvature of a Kähler metric can always unambiguously be understood as the Riemannian curvature.

Let R denote the Riemannian curvature tensor of a Hermitian metric g. We view R as the $(0,4)$-tensor $\mathrm{R} : (T^{\mathbf{R}}X)^{\otimes 4} \to \mathbf{R}$ given by

$$\mathrm{R}(u,v,w,z) := \mathrm{g}\left(\mathrm{D}_u\mathrm{D}_v w - \mathrm{D}_v\mathrm{D}_u w - \mathrm{D}_{[u,v]}w, z\right).$$

Extending D and R complex linearly to $T^{\mathbf{C}}X$. If g is a Kähler metric, then the only non-zero components of the curvature tensor are $\mathrm{R}(u,\bar{v},w,\bar{z})$ together with their symmetries

$$\mathrm{R}(u,\bar{v},w,\bar{z}) \;=\; -\mathrm{R}(\bar{v},u,w,\bar{z}) \;=\; \mathrm{R}(\bar{v},u,\bar{z},w),$$

and their conjugates

$$\overline{\mathrm{R}(u,\bar{v},w,\bar{z})} \;=\; \mathrm{R}(v,\bar{u},z,\bar{w}).$$

**Definition 12.** Let $(X,\mathrm{g})$ be a Kähler manifold. The *Ricci curvature* of g is the $(0,2)$-tensor

$$\begin{aligned}\mathrm{Ric}(\mathrm{g}) &:= \sum_{i,j} \mathrm{Ric}_{i\bar{j}}(\mathrm{d}z^i \otimes \mathrm{d}\bar{z}^j + \mathrm{d}\bar{z}^j \otimes \mathrm{d}z^i) \\ &= \sum_{i,j,k,\ell} \left(\mathrm{g}^{k\bar{\ell}}\mathrm{R}_{i\bar{j}k\bar{\ell}}\right)(\mathrm{d}z^i \otimes \mathrm{d}\bar{z}^j + \mathrm{d}\bar{z}^j \otimes \mathrm{d}z^i).\end{aligned}$$

The *Ricci form* is then the real $(1,1)$-form associated to $\mathrm{Ric}(\mathrm{g})$, i.e.,

$$\mathrm{Ric}(\omega) \;=\; \sqrt{-1}\sum_{i,j} \mathrm{Ric}_{i\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j.$$

**Cautionary Remark 1.10.** *In the same way that the $(1,1)$-form $\omega$ associated to a Hermitian metric g is sometimes referred to as a metric, the Ricci form and Ricci curvature are often spoken about interchangeably.*

The following theorem describing the local formula for the Ricci curvature of a Kähler metric can only be described as an act of divine intervention.

**Theorem 1.11.** *Let $(X,\omega)$ be a Kähler manifold. The Ricci curvature is locally given by*

$$Ric(\omega) \;=\; -\sqrt{-1}\partial\bar{\partial}\log\omega^n.$$

A proof of this result can be found in [56]. This local formula for the Ricci curvature comes from the fact that $\omega^n$ defines a Hermitian metric on $K_X$. A more detailed discussion is provided in the next section.

*Remark 6.* The formula for the Ricci curvature given in Theorem 1.11 also holds for non-Kähler Hermitian metrics. The meaning of 'Ricci curvature' is more delicate: The formula is valid for the 'first Chern Ricci curvature', the contraction of the third and fourth indices of the Chern curvature tensor (see, e.g., [11]).

**Definition 13.** Let $(X,\omega_0)$ be a compact Kähler manifold. The *normalized Kähler–Ricci flow* starting at $\omega_0$ is the family of Kähler metrics $\omega(t)$ satisfying

$$\frac{\partial\omega(t)}{\partial t} \;=\; -\mathrm{Ric}(\omega(t)) + \lambda\,\omega(t), \qquad \omega(t)|_{t=0} \;=\; \omega_0.$$

When $\lambda = 0$, this is called the *Kähler–Ricci flow*, and when referring to the normalized Kähler–Ricci flow explicitly, $\lambda = -1$.

The Kähler–Ricci flow is the Ricci flow starting from a Kähler metric. The fact that the Ricci flow preserves the Kähler condition is a consequence of the fact that it can be expressed in terms of the associated $(1,1)$-forms $\omega(t)$ and $\mathrm{Ric}(\omega(t))$.

*Remark 7.* For Hermitian non-Kähler metrics, the Ricci flow does not preserve the Hermitian structure. In place of the Ricci flow is the class of Hermitian curvature flows [58] (which includes the pluriclosed flow [57]), and the Chern Ricci flow [21, 69, 70].

## 2 Kähler–Einstein metrics

### *2.1 Ample and positive line bundles*

Recall that a line bundle $L \to X$ is a holomorphic vector bundle of rank one. The space of global holomorphic sections of $L$ is denoted by $H^0(X,L)$, and is a finite-dimensional vector space if $X$ is compact. In general, $L$ may have no global sections (e.g., the canonical bundle of $\mathbf{CP}^n$). In the event that $L$ has global sections $\sigma_0,\ldots,\sigma_N \in H^0(X,L)$, evaluating these sections at each point of $x \in X$, defines a holomorphic map

$$\Phi : X \to Y \subseteq \mathbf{CP}^N, \qquad \Phi(x) \;:=\; (\sigma_0(x),\ldots,\sigma_N(x)),$$

called the *Iitaka map* associated to $L$ (see, e.g., [23, 39]).

**Definition 14.** A line bundle $L \to X$ over a compact complex manifold $X$ is *ample* if there exists $m \in \mathbf{N}$ such that the Iitaka map associated to $mL$ is a holomorphic embedding.

*Remark 8.* It is immediate from Definition 14 that compact complex manifolds with an ample line bundle are projective (and hence, Kähler). Of course, not every compact Kähler is projective: There are non-projective complex tori and non-projective K3 surfaces.

*Example 4.* On $\mathbf{CP}^n = (\mathbf{C}^{n+1} \setminus \{0\})/\mathbf{C}^\star$, the *tautological bundle* is the line bundle $\mathcal{O}_{\mathbf{CP}^n}(-1)$ whose fiber at $x \in \mathbf{CP}^n$ is the corresponding line in $\mathbf{C}^{n+1}$. The *hyperplane bundle* $\mathcal{O}_{\mathbf{CP}^n}(1)$ is the line bundle dual to $\mathcal{O}_{\mathbf{CP}^n}(-1)$. The sections $H^0(\mathbf{CP}^n, \mathcal{O}_{\mathbf{CP}^n}(1))$ are linear forms, and the Iitaka map $\Phi : \mathbf{CP}^n \to \mathbf{CP}^n$ is the identity map. Hence, $\mathcal{O}_{\mathbf{CP}^n}(1)$ is ample.

It was discovered by Kodaira that ampleness of $L$ can be phrased in terms of curvature of Hermitian metrics on $L$.

**Definition 15.** Let $L \to X$ be a line bundle. A *Hermitian metric* $h$ on $L$ is a smooth family of Hermitian inner products on each fiber. The *curvature form* of $h$ is the $(1,1)$-form locally given by

$$\Theta(L,h) \;=\; \sqrt{-1}\partial\bar{\partial}\log(h).$$

We say that $L$ is *positive* (respectively, *flat*) if there is a Hermitian metric $h$ with positive (respectively, zero) curvature form.

*Remark 9.* A few clarifications are in order regarding Definition 15: The holomorphic variation in the fibers of a line bundle $L$ is encoded in a 'holomorphic structure' $\bar{\partial}$. Given a Hermitian metric $h$ on $L$, there is a unique connection $\nabla$ on $L$ such that $\nabla h = 0$ and $\nabla^{0,1} = \bar{\partial}$.

*Remark 10.* It is immediate from Definition 15 that the curvature form satisfies $\Theta(mL) = m\Theta(L)$ for any $m \in \mathbf{Z}$.

*Example 5.* Let $X$ be a complex manifold with $\dim_{\mathbf{C}} X = n$. Let $\alpha = f(z)\mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n$ be a local section of the canonical bundle $K_X$. A Hermitian metric $h$ on $K_X$ is given by

$$h(\alpha) = e^{-\varphi}\alpha \wedge \bar{\alpha} \;=\; e^{-\varphi}|f(z)|^2 \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n \wedge \mathrm{d}\bar{z}^1 \wedge \cdots \wedge \mathrm{d}\bar{z}^n.$$

In particular, a Hermitian metric on $K_X$ is equivalent to a volume form.

*Example 6.* The hyperplane bundle $\mathcal{O}_{\mathbf{CP}^n}(1)$ is positive. Indeed, over an affine chart, say, $U_0 := \{[z_0 : \cdots : z_n] \in \mathbf{CP}^n : z_0 = 1\} \simeq \mathbf{C}^n$, we can easily see that the curvature form of the Hermitian metric

$$h(z,\xi) := \frac{|\xi|^2}{1 + |z_1|^2 + \cdots + |z^n|^2}, \quad \xi \in \mathcal{O}_{\mathbf{CP}^n}(1)_{[z]} \simeq \mathbf{C},$$

is the Fubini–Study metric on $\mathbf{CP}^n$, and therefore, $\mathcal{O}_{\mathbf{CP}^n}(1)$ is positive.

*Example 7.* An ample line bundle is positive: If $\Phi : X \to Y$ is the Iitaka map, given by evaluating sections of $mL$, then $mL \simeq \Phi^* \mathcal{O}_{\mathbf{CP}^n}(1)$. Hence, by pulling back the metric whose curvature is the Fubini–Study metric, we get a metric on $L$ with positive curvature.

*Example 8.* The trivial line bundle $\mathcal{O}_X$ is flat. In particular, if $L$ is a torsion line bundle in the sense that $mL \simeq \mathcal{O}_X$, then $L$ is flat.

The following theorem of Kodaira shows that the converse to Example 7 is also true. The standard proof can be found in [23].

**Theorem 2.1.** *(Kodaira embedding theorem). A line bundle $L \to X$ over a compact complex manifold is ample if and only if it is positive.*

Combining Example 5, Example 8, Theorem 1.11, and Theorem 2.1, we have the following corollary.

**Corollary 1.** *Let $X$ be a compact complex manifold. If $X$ admits a Kähler metric with*

*(1) $Ric(\omega) > 0$, then $X$ is Fano in the sense that $-K_X$ is ample.*
*(2) $Ric(\omega) = 0$, then $X$ is Calabi–Yau in the sense that $K_X$ is torsion.*
*(3) $Ric(\omega) < 0$, then $X$ is canonically polarized in the sense that $K_X$ is ample.*

*In case (1) and case (3), $X$ is projective.*

*Remark 11.* We mentioned that there are complex tori $\mathbf{C}^n/\Lambda$ that are not projective. These are Calabi–Yau in the sense of Corollary 1. The simply connected Calabi–Yau surfaces are called *K3 surfaces* (or equivalently, a compact complex surface diffeomorphic to a quartic hypersurface in $\mathbf{CP}^3$). The moduli space of (marked) K3 surfaces is 20-dimensional (and connected). The projective K3 surfaces form a 19-dimensional subspace (see, e.g., [28]).

*Example 9. Enriques surfaces* are compact Kähler surfaces with $2K_X \simeq \mathcal{O}_X$ but $K_X \not\simeq \mathcal{O}_X$. Enriques surfaces are doubly covered by K3 surfaces.

The converse to Corollary 1 is the Calabi conjecture. Let $X$ be a compact Kähler manifold with $\dim_{\mathbf{C}} X = n$. From Example 7 and Example 5, if $K_X$ (or $-K_X$) is ample, then $X$ admits a volume form $\Omega \in \Lambda^{n,n}(X)$, such that $\Theta(\Omega) = \sqrt{-1}\partial\bar{\partial}\log(\Omega)$ is positive (respectively, negative). Yau's resolution of the Calabi conjecture [75] asserts that we can always find a Kähler metric $\omega$ whose volume form $\omega^n$ is precisely $\Omega$.

**Theorem 2.2.** *(The Calabi–Yau theorem). Let $X$ be a compact Kähler manifold. Then $X$ admits a Kähler metric with*

*(1) $Ric(\omega) > 0 \iff X$ is Fano.*
*(2) $Ric(\omega) = 0 \iff X$ is Calabi–Yau.*
*(3) $Ric(\omega) < 0 \iff X$ canonically polarized.*

*In case (2), there exists a unique Ricci-flat Kähler metric in each Kähler class. In case (3), there is a unique Kähler–Einstein metric with* $Ric(\omega) = -\omega$.

Statement (3) was also proved independently by Aubin [2].

*Example 10.* The only Fano curve is $\mathbf{CP}^1$. Fano surfaces are sometimes called *Del Pezzo surfaces*. All Fano surfaces are biholomorphic to $\mathbf{CP}^1 \times \mathbf{CP}^1$ or the blow-up of $\mathbf{CP}^2$ at $0 \leq k \leq 8$ points in general position. Here, *general position* means that no three points lie on a line, no six lie on a conic, and when $k = 8$, the points do not lie on a singular cubic curve with a singularity at one of the points.

*Remark 12.* In general, a Fano manifold need not admit a Kähler–Einstein metric with $\mathrm{Ric}(\omega) = \omega$. Matsushima [41] showed that the Lie algebra $\mathfrak{h}$ of holomorphic vector fields on a Kähler–Einstein Fano manifold is reductive, with $\mathfrak{h} \simeq \mathfrak{k} \otimes \mathbf{C}$ the complexification of the Lie algebra of Killing vector fields. For Fano surfaces, Matsushima's criterion is equivalent to the existence of a Kähler–Einstein metric [60].

Let $X_k := \mathbf{CP}^2 \sharp k \overline{\mathbf{CP}^2}$ denote the Fano surface given by blowing up $k$ points in general position. The Lie algebra of holomorphic vector fields on $X_k$ is reductive for $k \in \{0,3,4,5,6,7,8\}$. Tian–Yau [64] produced Kähler–Einstein metrics on $X_3$ and $X_4$. For $k > 4$, the Lie algebra of holomorphic vector fields on $X_k$ is trivial (see, e.g., [60, 66]).

## 2.2 Calabi–Yau manifolds

The terminology 'Calabi–Yau' is not entirely fixed in the literature. The definition adopted in Corollary 1 is one of the most general. When $X$ is not assumed to be Kähler, this is often emphasized with the prefix non-Kähler (see, e.g., [67]). The definition we have adopted in Corollary 1 is the one that coincides with the existence of a Ricci-flat Kähler metric (on compact Kähler manifolds). The discrepancy in the terminology is primarily a consequence of the following decomposition theorem due to Bogomolov [9, 8] and Beauville [5].

**Theorem 2.3.** *(Beauville–Bogomolov). Let $X$ be a Calabi–Yau manifold. The universal cover of $X$ then splits as*

$$\widetilde{X} \simeq \mathbf{C}^k \times \prod_i Y_i \times \prod_j Z_j,$$

*where*

*(1) the $Y_i$ are simply connected projective manifolds with $K_{Y_i} \simeq \mathcal{O}_{Y_i}$ and no intermediate holomorphic forms, i.e., $h^{p,0}(Y_i) = 0$ for $1 \leq p \leq \dim_{\mathbf{C}} Y_i - 1$;*

*(2) the $Z_j$ are simply connected projective manifolds with a unique (up to scale) holomorphic symplectic 2-form and no nontrivial holomorphic $p$-forms for odd $p$, i.e., $h^{0,p}(Z_j) = 0$ for odd $p$.*

The manifolds described in (i) are sometimes called *strict Calabi–Yau manifolds*, while the manifolds in (ii) are called *hyperkähler* or *irreducible holomorphically symplectic manifolds*. The proof of Theorem 2.3 is based on holonomy. In particular, the strict Calabi–Yau manifolds are precisely those with holonomy contained in $\mathsf{SU}(n)$, while the hyperkähler manifolds have holonomy contained in $\mathsf{Sp}(n)$.

*Remark 13.* In mirror symmetry, Calabi–Yau threefolds almost always refers to strict Calabi–Yau manifolds of (complex) dimension 3. By elementary arguments (see, e.g., [26]), the vanishing of $h^{1,0}$ and $h^{2,0}$ for a strict Calabi–Yau threefold forces the only variable Hodge numbers to be $h^{1,1}$ (dimension of the vector space containing the Kähler cone) and $h^{1,2}$ (dimension of the versal deformation space).

## 2.3 The first Chern class

Ampleness of a line bundle affords a characterization in terms of a cohomology class $c_1(L) \in H^2(X, \mathbf{Z})$ called the *first Chern class of L*. As we remarked in Remark 1.6, the line bundles form a group $\mathrm{Pic}(X)$. Because of the bundle charts defining the line bundle, $\mathrm{Pic}(X)$ is isomorphic to $H^1(X, \mathcal{O}_X^\star)$ (see [23]). From the exponential sequence of sheaves $0 \to \mathbf{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\star \to 0$, there is a long exact sequence on sheaf cohomology groups. The first boundary map $c_1 : \mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}_X^\star) \to H^2(X, \mathbf{Z})$ then maps a line bundle $L$ to its first Chern class $c_1(L) \in H^2(X, \mathbf{Z})$. Tensoring the cohomology groups with $\mathbf{R}$ kills the torsion in $H^2(X, \mathbf{Z})$ and we get a cohomology class $c_1^{\mathbf{R}}(L) \in H^2_{\mathrm{DR}}(X, \mathbf{R})$ represented by the curvature form of a Hermitian metric (see [23]).

**Theorem 2.4.** *Let $L \to X$ be a line bundle on $X$. The cohomology class represented by $\Theta(L,h) = \sqrt{-1}\partial\bar{\partial}\log(h)$ is independent of the choice of Hermitian metric h, and represents $2\pi c_1^{\mathbf{R}}(L)$.*

*Example 11.* Let $X$ be a compact Kähler manifold. Then $c_1^{\mathbf{R}}(K_X) = 0$ in $H^2_{\mathrm{DR}}(X, \mathbf{R})$ if and only if $mK_X \simeq \mathcal{O}_X$ for some $m \in \mathbf{N}$, i.e., $X$ is Calabi–Yau. Enriques surfaces [4] show that $K_X$ is not necessarily holomorphically trivial.

*Example 12.* From Theorem 2.1 and Theorem 2.4, a line bundle $L \to X$ is ample if and only if $c_1(L) > 0$, i.e., $c_1^{\mathbf{R}}(L) \in \mathcal{K}$. In particular, $X$ is canonically polarized (respectively, Fano) if and only if $c_1^{\mathbf{R}}(K_X) \in \mathcal{K}$ (respectively, $c_1^{\mathbf{R}}(-K_X) \in \mathcal{K}$).

*Remark 14.* The expression 'first Chern class of $X$' is often used. This is always understood to mean $c_1(-K_X)$. In particular, for compact Kähler manifolds, positive first Chern class is equivalent to the existence of a Kähler metric with positive Ricci curvature, and similarly, for negative and zero first Chern class [75].

## 3 The Analytic Minimal Model Program

The analytic minimal model program proposed by Song and the second named author [62, 53, 54, 55] aims to realize the minimal model program in birational algebraic geometry canonically using the Kähler–Ricci flow.

### *3.1 An overview of the MMP*

The minimal model program (MMP) is a classification scheme for projective varieties (or more generally, compact Kähler manifolds) up to bimeromorphism. There are essentially two aspects to the program:

(1) Establish, within the bimeromorphism class, the existence of a model $X$ for which the canonical bundle $K_X$ is nef, or that $X$ is a Mori fiber space;
(2) Show that if $K_X$ is nef, then $K_X$ is semi-ample.

A compact Kähler manifold with $K_X$ nef is called a *minimal model*. The condition that $K_X$ is nef implies that $X$ has an arbitrarily small positive upper bound on its Ricci curvature (see Definition 16). At the other extreme, a *Mori fiber space* is a holomorphic map $f : X \to Y$ onto a lower-dimensional variety $Y$ whose smooth fibers are Fano manifolds. Aspect (2) of the MMP is called the *abundance conjecture*. It asserts that all compact Kähler manifolds are (up to bimeromorphism) constructed out of three model geometries: Fano, Calabi–Yau, and canonically polarized varieties (see Conjecture 1).

The MMP proposes that there are a finite number of varieties $X_0, \ldots, X_N$ with $X = X_0$ together with bimeromorphic maps

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \longrightarrow \cdots \longrightarrow X_{N-1} \xrightarrow{f_{N-1}} X_N \tag{1}$$

such that $X_N$ is either a minimal model or a Mori fiber space.

*Remark 15.* In complex dimensions $\leq 2$, the MMP can be carried out within the category of smooth varieties. For curves, no bimeromorphic maps are needed, and for surfaces, the $f_k : X_k \to X_{k+1}$ are all given by blow downs that arrive either at a minimal model or a Mori fiber space. However, beginning in dimension three, it is not possible to stay entirely within the category of smooth varieties.

Both aspects of the MMP are known for compact Kähler surfaces (see, e.g., [4]). The existence of minimal models for projective threefolds was proven in a series of papers [43, 44, 49, 32, 33, 35, 52, 46, 37]. For compact Kähler threefolds, this was proven in [29]. For projective manifolds of general type, the existence of minimal models is proven in [6]. Abundance for projective threefolds was proven in [42, 34, 36], and extended to compact Kähler threefolds in [13, 14].

*Example 13.* The curves $C$ with $K_C$ nef are precisely those with genus $g(C) \geq 1$, while $\mathbf{CP}^1$ is the only Mori fiber space in dimension one. For compact Kähler surfaces $X$, minimal models are precisely those surfaces with no $(-1)$-curves (c.f., Theorem 1.8). In particular, minimal models cannot be blown down.

For curves, the minimal models and Mori fiber spaces were identified by considering the genus. By Riemann–Roch, the genus is equal to the dimension of the space of holomorphic sections of the canonical bundle,

$$g(C) = \dim H^0(C, K_C).$$

For surfaces, and in higher dimensions, the genus is replaced by the Kodaira dimension $\kappa(X)$, which measures the growth rate of the *plurigenera* $p_m(X) := \dim H^0(X, mK_X)$ as a function of $m \in \mathbf{N}$. The Kodaira dimension is a bimeromorphic invariant that is either $-\infty$, or a nonnegative integer $0 \leq \kappa(X) \leq \dim_{\mathbf{C}} X$.

*Example 14.* For curves, $\kappa(\mathbf{CP}^1) = -\infty$, $\kappa(\mathbf{C}/\Lambda) = 0$, and $\kappa(\mathbf{D}/\Gamma) = 1$. In particular, case (i) corresponds to $\kappa(X) \geq 0$ and case (ii) corresponds to $\kappa(X) = -\infty$.

*Remark 16.* The Kodaira dimension measures the proportions of the three model geometries required to build $X$ (up to bimeromorphism). For example, if $X$ is a minimal compact Kähler threefold, then $\kappa(X) = 0$ if and only if $X$ is a Calabi–Yau manifold; $\kappa(X) = 1 \iff X_N$ is fibered by Calabi–Yau surfaces over a hyperbolic curve; $\kappa(X) = 2 \iff X_N$ is fibered by elliptic curves over a surface of general type; $\kappa(X) = 3 \iff X_N$ is minimal and of general type.

## *3.2 The Kähler–Ricci flow and MMP*

The analytic minimal model program aims to establish the MMP for compact Kähler manifolds using the Kähler–Ricci flow, analogous to Hamilton's program for proving Thurston's geometrization conjecture using the Ricci flow. The analytic MMP was proposed by the second-named author and J. Song in [53, 54, 63]. Broadly, the program can be described as follows (c.f., [56]):

*Case 1.* If the Kähler–Ricci flow exhibits a finite-time singularity at time $T$, then the Kähler–Ricci flow deforms $(X, \omega_0)$ to $(Y, \mathrm{g}_Y)$ with a possibly singular metric $\mathrm{g}_Y$ as $t \to T$. If $\dim X = \dim Y$, and $Y$ differs from $X$ by a codimension one subvariety, then we start the Kähler–Ricci flow on $(Y, \mathrm{g}_Y)$. If $\dim X = \dim Y$ and $Y$ differs from $X$ by a subvariety of higher codimension, then $Y$ will be singular. By considering an appropriate notion of weak Kähler–Ricci flow on $Y$, starting at $\mathrm{g}_Y$, the flow should immediately resolve the singularities of $Y$ and replace $Y$ by its *flip*[4] $X^+$. We now run the Kähler–Ricci flow starting on $X^+$. If $0 < \dim Y < \dim X$, we run the Kähler–Ricci flow on $(Y, \mathrm{g}_Y)$. If $\dim Y = 0$, then $X$ should be Fano. After appropriate

[4] We have chosen to avoid defining what flips are. The reader is encouraged to see [56, page 81].

normalization, the Kähler–Ricci flow should deform to $(X', \omega')$ where $X'$ is possibly a different manifold and $\omega'$ is either a Kähler–Einstein metric or a Kähler–Ricci soliton. Assuming the existence of flips, a weak solution for the Kähler–Ricci flow through finite time singularities is constructed in [55]. The weak solution is smooth outside the singularities of $X$ and the exceptional locus of the contractions and flips. It is a nonnegative closed $(1,1)$-current with locally bounded potentials and unique. The most recent progress in this case appears in [31].

*Case 2.* If the Kähler–Ricci flow exists for all time, then $K_X$ is nef, and thus, $X$ is a minimal model. If $\kappa(X) = 0$, then the Kähler–Ricci flow converges to the unique Ricci-flat Kähler metric in the Kähler class of $\omega_0$. If $\kappa(X) = \dim_{\mathbf{C}} X$, then the Kähler–Ricci flow is *non-collapsed*, and converges to a singular Kähler–Einstein metric (see Section 3.5). If $0 < \kappa(X) < \dim_{\mathbf{C}} X$, then the Kähler–Ricci flow exhibits *volume collapsing* and converges to a singular generalized Kähler–Einstein metric (see Section 3.6).

## *3.3 Nef line bundles*

The Kähler cone $\mathcal{K}$ of a compact Kähler manifold $X$ is an open convex cone in the finite-dimensional vector space $H^{1,1}(X, \mathbf{R})$. A line bundle $L \to X$ is nef if $c_1(L) \in \overline{\mathcal{K}}$. For a general complex manifold, a nef line bundle is defined as follows [17].

**Definition 16.** A line bundle $L \to X$ is *nef* if for any $\varepsilon > 0$ and any positive $(1,1)$-form $\omega_0$ there exists a smooth Hermitian metric $h$ on $L$ with $\Theta(L,h) \geq -\varepsilon\omega_0$. A compact Kähler manifold with $K_X$ nef is called a *minimal model*.

It is here that the Kähler–Ricci flow provides the most insight. Indeed, the following of Tian–Zhang [65] provides an equivalent description of nef canonical bundle for compact Kähler manifolds.

**Theorem 3.1.** *Let $X$ be a compact Kähler manifold. The Kähler–Ricci flow, starting from an initial metric $\omega_0$ exists for $t \in [0, T)$, where*

$$T := \sup\{t > 0 : \{\omega_0\} - 2\pi t\{Ric(\omega_0)\} \in \mathcal{K}\}.$$

*In particular, the Kähler–Ricci flow exists for all time (starting from any initial Kähler metric) if and only if the canonical bundle $K_X$ is nef.*

*Example 15.*

(i) Canonically polarized and Calabi–Yau manifolds have nef canonical bundle.
(ii) If $X$ is projective with no rational curves, i.e., every holomorphic map $\mathbf{CP}^1 \to X$ is constant, then $K_X$ is nef [45]. The same statement holds for compact Kähler manifolds up to dimension 3, see [29].
(iii) Compact Kähler manifolds with Kähler (or more generally, pluriclosed) metrics of non-positive holomorphic sectional curvature have nef canonical bundle [71, 11].

### *3.4 The Kodaira dimension* $\kappa(X)$

While the condition that $K_X$ is nef can often be verified in practice, and also informs us of the behavior of the Kähler–Ricci flow, it does not provide much structural information about the manifold. However, the (Kähler extension of the) abundance conjecture predicts that for compact Kähler manifolds, nef canonical bundle is equivalent to the stronger notion of semi-ampleness. Before presenting the abundance conjecture and definition of semi-ample, we define the Kodaira dimension.

**Definition 17.** Let $X$ be a compact complex manifold. The *Kodaira dimension* $\kappa(X)$ is defined by

$$\kappa(X) := \limsup_{m\to\infty} \frac{\log \dim_{\mathbf{C}} H^0(X, mK_X)}{\log(m)}$$

if $H^0(X, mK_X) \neq \{0\}$ for some $m \in \mathbf{N}$. If $H^0(X, mK_X) = \{0\}$ for all $m \in \mathbf{N}$, we declare $\kappa(X) = -\infty$.

The fundamental facts about the Kodaira dimension are enumerated in the following theorem.

**Theorem 3.2.** *Let $X$ be a compact complex manifold. The Kodaira dimension $\kappa(X)$ is a bimeromorphic invariant with $\kappa(X) \in \{-\infty, 0, 1, ..., \dim_{\mathbf{C}} X\}$. If $X$ is compact Kähler with $K_X$ nef, then $\kappa(X) \geq 0$.*

*Example 16.*

(i) Compact Kähler manifolds that are uniruled in the sense that they are covered by rational curves, have $\kappa(X) = -\infty$. The uniruled conjecture predicts that the converse is true.

(ii) If $X$ is Calabi–Yau, then $\kappa(X) = 0$. The abundance conjecture (see Conjecture 1) implies the converse.

(iii) If $0 < \kappa(X) < \dim_{\mathbf{C}} X$, the Iitaka map defines a surjective holomorphic map $\Phi : X \to Y$ onto a normal irreducible and reduced projective variety with connected fibers. The smooth fibers $F$ of $\Phi$ have $\kappa(F) = 0$.

(iv) If $\kappa(X) = \dim_{\mathbf{C}} X$, then $X$ is said to be of *general type*. Compact Kähler manifolds of general type are projective. Moreover, if $X$ is of general type but not canonically polarized, then $X$ has a rational curve.

**Definition 18.** Let $X$ be a compact Kähler manifold. The canonical bundle $K_X$ is *semi-ample* if there exists $m \in \mathbf{N}$ and sections $\sigma_0, ..., \sigma_N \in H^0(X, mK_X)$ such that the map

$$\Phi : X \longrightarrow Y \subseteq \mathbf{P}(H^0(X, mK_X)), \qquad \Phi(x) := (\sigma_0(x), ..., \sigma_N(x))$$

is a surjective holomorphic map with connected fibers.

The (Kähler extension of the) abundance conjecture, however, asserts that $K_X$ is nef if and only if $K_X$ is semi-ample.

*Conjecture 1.* (Abundance). The canonical bundle of a minimal model is semi-ample. In particular, there is a surjective holomorphic map $f : X \to Y$ with connected fibers onto a mildly singular projective variety $Y$ such that the smooth fibers are Calabi–Yau. Moreover, $\dim_{\mathbf{C}} Y = \kappa(X)$.

Conjecture 1 is known for compact Kähler manifolds up to dimension 3.

*Remark 17.* Conjecture 1 is specific to $K_X$, and is false for general holomorphic line bundles. If $L \to X$ is a semi-ample line bundle, then $L$ is nef. In general, a nef line bundle need not be semi-ample. An example due to Serre is described in [10].

*Remark 18.* Conjecture 1 also implies that if $K_X$ is nef, then $K_X$ admits a holomorphic section, and therefore, $\kappa(X) \geq 0$. For compact non-Kähler manifolds, this does not hold: Inoue surfaces are non-Kähler compact complex surfaces with nef canonical bundle and $\kappa(X) = -\infty$ (see, e.g., [4]).

*Remark 19.* The singularities of the base variety $Y$ in Conjecture 1 are referred to as 'canonical singularities' and occur in (complex) codimension at least two (see [38, 50]).

## *3.5 Non-collapsing Kähler–Ricci flows*

Assume the abundance conjecture holds. If $X$ is a compact Kähler manifold with $K_X$ nef, the normalized Kähler–Ricci flow starting from any initial metric $\omega_0$ exhibits volume collapsing if and only if $0 < \kappa(X) < \dim_{\mathbf{C}} X$. If $\kappa(X) = 0$, Conjecture 1 implies that $X$ is Calabi–Yau, and the behavior of the Kähler–Ricci flow is described by the following theorem [15].

**Theorem 3.3.** *If $X$ is Calabi–Yau, then the Kähler–Ricci flow converges smoothly to the unique Ricci-flat Kähler metric in the Kähler class of the initial metric.*

The other extreme case occurs when $\kappa(X) = \dim_{\mathbf{C}} X$, i.e., $X$ is of general type. In this case, the behavior of the normalized Kähler–Ricci flow is understood from the following theorems of Tian–Zhang [65] and Tsuji [72].

**Theorem 3.4.** *Let $X$ be a compact Kähler manifold with $K_X$ nef and $\kappa(X) = \dim_{\mathbf{C}} X$. There exists a proper closed analytic subvariety $Z \subset X$ and a Kähler metric $\omega_\infty$ on $X \setminus Z$ such that $Ric(\omega_\infty) = -\omega_\infty$ on $X \setminus Z$. Given any Kähler metric $\omega_0$ on $X$, the normalized Kähler–Ricci flow starting from $\omega_0$ converges to $\omega_\infty$ locally smoothly on $X \setminus Z$ as $t \to \infty$.*

The subvariety $Z$ can be described explicitly as the union of subvarieties $V \subset X$ such that $\int_V (-c_1(X))^{\dim(V)} = 0$. This subvariety $Z$ is empty if and only if $X$ is canonically polarized. In this case, $\omega_\infty$ is the Kähler–Einstein metric obtained from the Aubin–Yau theorem.

## *3.6 Collapsing Kähler–Ricci flows*

Let $X$ be a compact Kähler manifold with semi-ample canonical bundle and intermediate Kodaira dimension $0 < \kappa(X) < \dim_{\mathbf{C}} X$. The Iitaka map $\Phi : X \to Y$ has positive-dimensional fibers, the smooth fibers are Calabi–Yau manifolds. In general, the singularities come from two sources: the singularities of $Y$ (which occur in codimension at least two, since $Y$ is normal) and the additional points where $\Phi$ fails to be a holomorphic submersion. The locus of points in $y \in Y$ such that the fiber $\Phi^{-1}(y)$ is singular, is commonly referred to as the *discriminant locus*.

*Example 17.* Let $X$ be a compact Kähler surface with nef $K_X$ and $\kappa(X) = 1$. The base $Y$ of the Iitaka map is then a smooth (normal varieties are smooth in dimension one) curve of genus $g \geq 1$ and the smooth fibers are elliptic curves (i.e., complex tori $\mathbf{C}/\Lambda$). The complex structure on an elliptic curve is determined by the $j$-invariant. The variation in the complex structure of the fibers of the Iitaka map is described by a holomorphic map $j : Y \setminus \mathrm{Disc}(\Phi) \to \mathbf{H}/\mathsf{SL}(2,\mathbf{Z})$, where $\mathbf{H}/\mathsf{SL}(2,\mathbf{Z})$ is the moduli space of elliptic curves. If all fibers of $\Phi$ are smooth, i.e., $\mathrm{Disc}(\Phi) = \varnothing$, then $j$ is a holomorphic function on a compact manifold, and is therefore constant.

There is a proper closed analytic subvariety $D \subset Y$ such that $Y^\circ := Y \setminus D$ is smooth and does not contain any critical values of $\Phi$. Hence, if we set $X^\circ := X \setminus \Phi^{-1}(D)$, then $\Phi : X^\circ \to Y^\circ$ is a proper holomorphic submersion, and thus a smooth fiber bundle by Ehresmann's theorem. The smooth fibers $\Phi^{-1}(y)$, for $y \in Y^\circ$ have $\kappa(\Phi^{-1}(y)) = 0$, and thus, Conjecture 1 implies that the smooth fibers are Calabi–Yau manifolds.

By solving a complex Monge–Ampère equation on $Y$, Song–Tian [54] showed that there exists a Kähler metric $\omega_{\mathrm{can}}$ on $Y^\circ$ of the form $\omega_{\mathrm{can}} = \omega_Y + \sqrt{-1}\partial\bar{\partial}u$ satisfying the twisted Kähler–Einstein equation

$$\mathrm{Ric}(\omega_{\mathrm{can}}) \;=\; -\omega_{\mathrm{can}} + \omega_{\mathrm{WP}},$$

where $\omega_{\mathrm{WP}}$ is the Weil–Petersson metric measuring the variation in the complex structure of the smooth Calabi–Yau fibers and $\omega_Y$ is a fixed background Kähler metric.

The following theorem is a combination of the results of [54, 25].

**Theorem 3.5.** *Let $(X, \omega_0)$ be a compact Kähler manifold with $K_X$ semi-ample and $0 < \kappa(X) < \dim_{\mathbf{C}} X$. The normalized Kähler–Ricci flow $\frac{\partial}{\partial t}\omega(t) = -Ric(\omega(t)) - \omega(t)$, $\omega(0) = \omega_0$, converges to $\Phi^*\omega_{can}$ in the sense of currents. Further,*

*(i) $\omega(t) \to \Phi^*\omega_{can}$ locally smoothly on $X^\circ$ in the sense that for any compact set $K \Subset X^\circ$ and $k \geq 0$, we have $\|\omega(t) - \Phi^*\omega_{can}\|_{\mathcal{C}^k(K,\omega_0)} \to 0$ as $t \to \infty$.*
*(ii) For any $K \Subset X^\circ$, there is $C > 0$ such that for all $t \geq 0$, we have*

$$\sup_K |Ric(\omega(t))|_{\omega(t)} \leq C.$$

## 4 Further directions and open problems

In this section, we discuss what is known about the Kähler–Ricci flow on compact Kähler manifolds with $K_X$ nef, without invoking Conjecture 1. The reader is also encouraged to see [55, 68, 76]. The Kodaira dimension $\kappa(X)$ measures the growth rate of sections of the pluricanonical bundles $mK_X$. However, it remains unknown whether nefness of $K_X$ implies that $mK_X$ has any sections for some $m \in \mathbf{N}$. Equivalently, in the absence of abundance it is unknown whether nef $K_X$ forces $\kappa(X) \geq 0$. Consequently, in this context one often replaces the Kodaira dimension by the numerical dimension.

**Definition 19.** Let $(X^n, \omega)$ be a compact Kähler manifold with $K_X$ nef. The *numerical dimension* $\nu(K_X)$ is the nonnegative integer $0 \leq k \leq n$, defined by

$$\nu(K_X) := \max\left\{k : \int_X c_1(K_X)^k \wedge \omega^{n-k}\right\}$$

If $K_X$ is nef and $\nu(K_X) = \dim_{\mathbf{C}} X$, then $K_X$ is big, and the results of Section 3.5 apply. The other extreme occurs when $K_X$ is nef and $\nu(K_X) = 0$, in which case $mK_X \simeq \mathcal{O}X$ (for some $m \in \mathbf{N}$) and Theorem 3.3 applies. Hence it suffices to consider the case $0 < \nu(K_X) < \dim \mathbf{C}X$.

The normalized Kähler–Ricci flow

$$\frac{\partial \omega(t)}{\partial t} = -\mathrm{Ric}(\omega(t)) - \omega(t), \qquad \omega(0) = \omega_0,$$

is equivalent to $\varphi(t)$ solving the parabolic complex Monge–Ampère equation

$$\begin{aligned}\frac{\partial \varphi(t)}{\partial t} &= \log \frac{e^{(n-\nu(K_X))t}\omega(t)^n}{\omega_0^n} - \varphi(t), \qquad \varphi(0) = 0,\\ \omega(t) &= e^{-t}\omega_0 - (1-e^{-t})\mathrm{Ric}(\omega_0) + \sqrt{-1}\partial\bar{\partial}\varphi_t > 0.\end{aligned}$$

For the remainder of this section, we assume that $(X^n, \omega_0)$ is a compact Kähler manifold with $K_X$ nef and $0 < \nu(K_X) < n$. Moreover, $\omega(t)$ refers to a solution of the normalized Kähler–Ricci flow. We also write $h$ for the Hermitian metric $h$ on $K_X$ induced from the initial metric $\omega_0$ with current form $\Theta(h) = -\mathrm{Ric}(\omega_0)$.

Based on the results of Fu–Guo–Song [19], Tosatti [68] proved the following[5] (extending results that had appeared earlier in [63] under the assumption that $K_X$ is semi-ample, cf., [63, page 88]).

**Theorem 4.1.** *For any sequence $t_k \to \infty$, there is a subsequence and a quasi-plurisubharmonic function $\varphi_\infty : X \to \mathbf{R} \cup \{-\infty\}$ such that $\chi_\infty := -Ric(\omega_0) + \sqrt{-1}\partial\bar{\partial}\varphi_\infty$ is a nonnegative $(1,1)$-current such that $\varphi(t_k) \to \varphi_\infty$ in $L^1(X)$ as $t \to \infty$. In particular, $\omega(t_k) \to \chi_\infty$ weakly in the sense of currents.*

[5] Tosatti proves a more general statement in [68], which we omit in order to avoid having to introduce further notions.

A priori, the subsequential limit $\varphi_\infty$ in Theorem 4.1 may depend on the choice of subsequence, and $\chi_\infty$ may depend on the initial metric $\omega_0$. The following conjecture was proposed in [68] (cf. [76]).

*Conjecture 2.* The subsequential limits $\varphi_\infty$ in Theorem 4.1 are unique, independent of the subsequence, and the current $\chi_\infty$ should be independent of the initial metric.

A resolution of Conjecture 2 implies that $he^{-\varphi_\infty}$ defines a singular metric on $K_X$ with semipositive curvature current $\chi_\infty$ that depends only on the complex structure of $X$. This singular metric can be used for $L^2$ methods, especially if further regularity properties can be established. When $K_X$ is semi-ample, $\varphi_\infty$ is continuous on $X$ (see [18]). Hence, a specific case of the abundance conjecture is the following.

*Conjecture 3.* In Conjecture 2, the limit potential $\varphi_\infty$ is continuous.

We also invite the reader to consult [68, 76] for further problems.

## 5 Relations among positivity notions

In this last section, we offer the reader a reference for the various positivity notions and the relations among them. For a holomorphic line bundle $L \to X$ on a compact Kähler manifold,

$$L \text{ positive} \iff L \text{ ample} \implies L \text{ semi-ample} \implies L \text{ nef}.$$

For general $L$, the second and third implications are not reversible. When $L = K_X$, we have

$$\begin{aligned} X \text{ canonically polarized} \iff K_X \text{ positive} &\iff c_1(K_X) > 0 \\ &\iff \exists!\, \mathrm{Ric}(\omega) = -\omega \\ &\iff K_X \text{ ample} \\ &\implies K_X \text{ semi-ample} \implies K_X \text{ nef}. \end{aligned}$$

Conjecture 1 implies that the last implication is reversible, i.e., $K_X$ semi-ample $\iff$ $K_X$ nef. For Calabi–Yau manifolds, we have

$$X \text{ Calabi–Yau} \iff c_1^{\mathbf{R}}(K_X) = 0 \iff mK_X \simeq \mathcal{O}_X \iff \mathrm{Ric}(\omega) = 0 \implies \kappa(X) = 0.$$

Again, Conjecture 1 implies that $\kappa(X) = 0 \iff mK_X \simeq \mathcal{O}_X$ follows from Conjecture 1. At the other extreme,

$$X \text{ general type} \iff K_X \text{ big} \iff \kappa(X) = \dim_{\mathbf{C}} X.$$

The following diagram summarizes the relationships between some of the notions discussed in previous sections. Note that we assume the validity of the abundance

conjecture for the implication $K_X$ nef $\implies$ $K_X$ semi-ample, and $\kappa(X) = 0 \implies \mathrm{Ric}(\omega) = 0$.

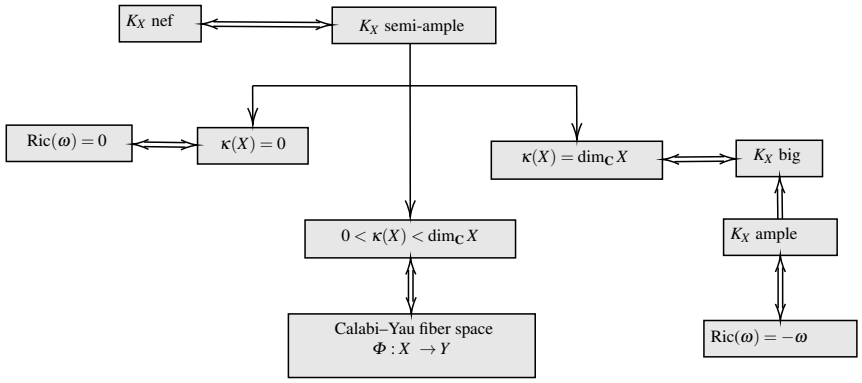


Diagram of implications for compact Kähler manifolds (assuming abundance).

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