Poisson processes.

The current workshop is devoted to rare events in dynamics and the aim of my minicourse is to discuss several problems there the statements do not involve rare events but the proofs relies crucially on the rare event theory.

However, before discussing these examples I would like to provide some background on Poisson processes.

Historically, Poisson limit theorem was among first results concerning rare events.

Recall that a random variable X has Poisson distribution with parameter λ denoted $X \sim Pois(\lambda)$ if $P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}$

Below I recall the basic properties of Poisson random variables which should be familiar from an introductory probability course and can be verified by direct computations.

(1) The generating function

$$M_X(z) := E(z^X) = \exp(\lambda(z-1))$$

(2) Consequently the characteristic function is

$$\varphi_X(t) = E(e^{itX}) = \exp(\lambda(e^{it}-1))$$

(3) If X_1, X_2 are independent $X_j \sim Pois(\lambda_j)$ then $X = X_1 + X_2 \sim Pois(\lambda_1+\lambda_2)$

(4) If X, X_1, X_2 are as in (3) then conditionally on X, X_i have binomial distribution

$$P(X_1 = k \mid X_1 + X_2 = n) = \binom{n}{k} p^k (1-p)^{n-k}$$

where $p = \frac{\lambda_1}{\lambda_1 + \lambda_2}$

(5) (Pruning) If $X \sim Pois(\lambda)$ and each point from X is removed with probability p independently of the others then the resulting random variable Y has Poisson distribution with parameter (λp)

(6) Factorial moments

$$P(X(X-1)\dots(X-k+1)) = \lambda^k.$$

~~[illegible]~~

Excercise 1 Prove properties (1) – (6).

Next we discuss Poisson processes.

~~[illegible]~~ Let $(\mathcal{X}, \mu)$ be a probability space. Let $\xi_1, \xi_2 \dots \xi_n$ be a random sequence of points in $\mathcal{X}$. We say that $\{\xi_n\}$ forms a Poisson process with parameters (X, μ) if denoting by $N(A)$ the number of points in A we have the following properties

(A) $N(A) \sim Pois(\mu(A))$

(B) $N(A_1)$ and $N(A_2)$ are independent if A_1 and A_2 are disjoint

Note that this definition is consistent due to property (B) of the Poisson process. We now discuss several properties of the Poisson process

(PP1) If $\xi \sim Pois(\mathcal{X}, \mu)$ and $\pi: \mathcal{X} \to \mathcal{Y}$ is a measurable map then

$\pi(\xi) \sim Pois(\mathcal{Y}, \pi^* \mu)$

Indeed if $B \subset \mathcal{Y}$ and $N(B)$ is the number of points from $\pi\xi$ in B then

$$P(N(B) = k) = P(N(\pi^{-1}(B)) = k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{where} \quad \lambda = \mu(\pi^{-1}(B)) = \pi^* \mu(B)$$

(PP2) If $\xi' \sim Pois(\mathcal{X}, \mu')$, $\xi'' \sim Pois(\mathcal{X}, \mu'')$ and ξ', ξ'' are independent then $\xi' \vee \xi'' \sim Pois(\mathcal{X}, \mu' + \mu'')$

<u>Exercise 2</u> Deduce (PP2) from property 3 of the Poisson random variable.

(PP3) Suppose that $\xi \sim Pois(\mathcal{X}, \mu)$ and suppose that we color each point from ξ in one of k colors so that for each point color j is chosen with probability p_j independently of the others. Then if $\xi^{(j)}$ are points of color j, then

$\xi^{(1)}, \xi^{(2)} \dots \xi^{(k)}$ are independent poisson processes

with parameters $(\mathcal{X}, p_j\mu)$.

<u>Exercise 3</u> Prove (PP3) (cf. property (5) of the Poisson random variable).

(PP4) If $\{\xi_n\} \sim \text{Poisson}(\mathcal{X}, \mu)$ and Y_n are independent identically distributed $\mathcal{Y}$ valued random variables with law P

then $\{(\xi_n, Y_n)\} \sim \text{Pois}(\mathcal{X} \times \mathcal{Y}, \mu \times P)$

<u>Exercise 4</u> Derive (PP4) from (PP3) in the case Y_n take values $\{1 \dots k\}$ so that $P(Y_n = j) = p_j$

(PP5) Conditioned on $N(A) = n$ the points in A are independent and distributed according to $P_A = \frac{\mu}{\mu(A)}$

That is $P(N(B) = k \mid N(A) = n) = \binom{n}{k} p^k (1-p)^{n-k}$ where $p = \frac{\mu(B)}{\mu(A)}$ for $B \subset A$.

<u>Exercise 5</u> Derive (PP5) from property (4) of the Poisson random variable.

(PP6) (Marker formula) Let $\Psi : \mathcal{X} \to \mathbb{R}$ be a measurable function and

$$S = \sum_{\xi} \Psi(\xi). \text{ Then}$$

$$E(e^{itS}) = \exp\left[\int_{\mathcal{X}} \left[e^{it\Psi(x)} - 1 \right] d\mu(x) \right]$$

Exercise 6 Prove (PP6)

Hint Approximate Ψ by piecewise constant functions.

One application of Poisson processes appears in relation to stable laws. We recall the corresponding facts. We say that a random variable $\mathcal{U}$ has stable law if for each $n \in \mathbb{N}$ $\exists\, a_n, c_n$ s.t. if $\mathcal{U}_1, \mathcal{U}_2 \ldots \mathcal{U}_n$ are independent identically distributed (iid) random variable having the same law as $\mathcal{U}$ then $\mathcal{U}_1 + \mathcal{U}_2 + \ldots + \mathcal{U}_n \overset{law}{=} a_n \mathcal{U} + c_n$

One can show that in this case $a_n = n^{1/\alpha}$ where $\alpha \in [0,2]$ is called the index of $\mathcal{U}$.

Classification of the stable laws has been accomplished by Levi and Khintchine. For $s=2$, $\mathcal{U}$ is normal while for $s \neq 2$,

~~[illegible]~~ $\mathcal{U} = \alpha_+ \mathcal{U}_+ - \alpha_- \mathcal{U}_- + \alpha_0$ where

$\alpha_{+1}, \alpha_{-1}, \alpha_0$ are are constants

$\alpha_\pm \geq 0$ and $\mathcal{U}_+$ and $\mathcal{U}_-$ are independent and

$$E(e^{it\mathcal{U}_\pm}) = \exp\left(\int_0^1 \left[e^{itu} - 1 - itu\right] \frac{s\,du}{u^{s+1}} + \int_1^\infty (e^{itu} - 1) \frac{s\,du}{u^{s+1}} \right)$$

Note that that for $s \in (0,1)$

$\mathcal{U}_\pm$ have the law of $\sum_n \xi_n$ up to normalization

where $\{\xi_n\} \sim \text{Pois}(\mathbb{R}_+, \mu_s)$

$\frac{d\mu(u)}{du} = \frac{s}{u^{s+1}}$ while for $s \in (1, 2)$ it is

the limit $\lim_{\varepsilon \to 0} \left(\sum_n \xi_n^\varepsilon - c_\varepsilon\right)$ where $\xi_n^\varepsilon \sim \text{Pois}((\varepsilon, \infty), \mu_s)$

and $c_\varepsilon = \varepsilon^{-s}$.

Exercise 7 For $s \in (0,1)$ show that $\sum_n \xi_n$ is stable using (PP1) and (PP2).

Exercise 8 Let $\{\xi_n\} \sim \text{Pois}(\mathbb{R}, \text{Leb})$. Let θ_n be iid symmetric with compact support. For $\alpha > 1/2$ show that $\mathcal{U} = \sum_n \frac{\theta_n}{\xi_n^\alpha}$ has stable law of index $1/\alpha$. (and $\mathcal{U}$ is symmetric)

Next we discuss proving the Poisson limit theorem for dynamical systems.

We say that a dynamical system $f : M \to M$ preserving a smooth measure μ is exponentially mixing of order r on $C^m(M)$ if

$$\left| \int \prod_{j=1}^{r} A_j(f^{n_j} x)\, d\mu(x) - \prod_{j=1}^{r} \mu(A_j) \right| \le K_{m,r}\, e^{-\beta L} \prod_{j=1}^{r} \|A_j\|_{C^m(M)}$$

where β is a positive constant and

$$L = \min_{i \neq j} |n_i - n_j|.$$

<u>Exercise 9</u> Show that $\forall m', m'' > 0$ $\circledast$ f is exponentially mixing of order r on $C^{m'}(M) \Leftrightarrow f$ is exponentially mixing of order r on $C^{m''}(M)$.

This justifies dropping $C^m(M)$ at the end and saying in the above scenario that f is exponentially mixing of order r.

<u>Thm 1</u> (B. Fayad – S. Liu – D.) If f is exponentially mixing of all orders then for almost every x_0 the following holds.

Let $S_\rho(x) = \sum_{k=0}^{N_\rho} 1_{B(x_0,\rho)}(f^k x)$ with $N_\rho^{\circledast} = \frac{2}{\mu(B(x_0, r))}$

Then if $\rho \to 0$ then $S_\rho \Rightarrow Pois(2)$.

Remark In all known examples of exponentially mixing systems one has a stronger result

$S_r \Rightarrow Pois(2)$ provided that x_0 is aperiodic.

However our proof does not give this if we only assume multiple exponential mixing.

Thus the description of exceptional points in Theorem 1 remains an interesting open problem.

The proof of Thm 1 consists of two parts.

We say that x_0 is slowly recurrent if $\forall A, K$ $\exists r_0$ such that for $r \le r_0$ we have

$$\mu(B(x_0,r) \cap f^{-n} B(x_0,r)) \le \frac{\mu(B(x_0,r))}{|\ln r|^A}$$

for all $n \le K|\ln r|$.

Prop 2 If x_0 slowly recurrent then $S_r \Rightarrow Pois(2)$

Prop 3 Almost every $x_0 \in \mathcal{M}$ is slowly recurrent

We start with Prop. 2. We use the following expression for the factorial moments of the number of visits

Let $S = \sum_{k=1}^{N} 1_{A_K}$. Then

$$E\left(\binom{S}{r}\right) = E\left[\frac{S(S-1)\dots(S-r+1)}{r!}\right] = \sum_{n_1 < n_2 < \dots < n_r} P(A_{n_1} A_{n_2} \dots A_{n_r})$$

Exercise 10 Multiple exponential mixing implies multiple exponential mixing on balls, that is

$\exists\, \varkappa, \beta$ s.t.

$$\left|\mu\left(\prod_{k=1}^{r} 1_{B(x_0,\rho)}(f^{n_k}x)\right) - \mu^r(B(x_0,\rho))\right| \le C\left(\frac{1}{r}\right)^{\varkappa} e^{-\beta L} \qquad L = \min_{i\neq j}|n_i - n_j|$$

We now show that

$$E(S_\rho(S_\rho - 1)\dots(S_\rho - r+1)) \xrightarrow[\rho\to 0]{} \lambda^r$$

We use (A). Take a constant $R \gg 1$. We claim that the main contribution to $(\hat{\Sigma})$ comes from the well separated tuples, that is, the tuples s.t.

$\min_{i\neq j}|n_i - n_j| > R|\ln r|.$

For well separated tuples

$$\mu\left(\prod_{j=1}^{r} 1_{B(x_0,r)}(f^{n_j}x)\right) = \mu(B(x_0,\rho))(1+\bar{\bar{o}}(1))$$

Since the number of tuples is $\frac{N^r}{r!}(1+\bar{\bar{o}}(1))$ we conclude that the contribution of well separated tuples to $E\left(\frac{S(S-1)\dots(S-r+1)}{r!}\right)$ is $\lambda^r(1+\bar{\bar{o}}(1))$.

It remains to control the tuples which are not well separated.

By cluster decomposition we mean partition

$\{n_1, n_2 \dots n_r\} = \bigcup_{j=1}^{s} C_j$ where the clusters $C_1 \dots C_s$ satisfy

(a) $\mathcal{D} := \max_{n', n'' \in C_j} |n' - n''| \leq \mathcal{K}(R, r) \ln \rho$.

(b) $\min_{\substack{n', n'' \text{ in different} \\ \text{clusters}}} |n' - n''| > R \max(\mathcal{D}, |\ln r|)$

Proceeding recursively starting from the partion $\{n_1\} \cup \{n_2\} \dots \cup \{n_r\}$ we see that each tupple admits cluster partition and

$s = r$ iff the tuple is well separated.

Let Ψ_ρ be a function so that

$\Psi_\rho(x) = 1$ if $x \in \mathcal{B}(x_0, \rho)$

$\Psi_\rho(X) = 0$ if $x \notin \mathcal{B}(x_0, 2\rho)$

and $\|\Psi\|_{C^1} \leq K/\rho$.

By definition of cluster partition

$$\mu\left(\prod_{i=1}^{r} 1_{\mathcal{B}(x_0,\rho)}(f^{n_j}x)\right) \leq \mu\left(\prod_{j=1}^{r} \Psi_\rho(f^{n_j}x)\right)$$

$$= \prod_{j=1}^{s} \mu(\varphi_{C_j}) + O(N^{-10r})$$

where $\varphi_{C_j} = \prod_{k \in C_j} \varphi_\rho(f^{n_k}x)$

Note that if C_j has size 1 then
$\mu(\mathcal{C}_{C_j}) \le \mu(B(x_0, 2\rho)) \le K\rho^d$ where
d is the dimension of the phase space while if
the size of C_j is greater than 1 then
$$\mu(\mathcal{C}_{C_j}) \le \mu(B(x_0, 2\rho) \cap f^{-\ell} B(x_0, 2\rho))$$
$$\le \frac{K\rho^d}{|\ln\rho|^{100r}}$$
by definition of slow recurrence where $\ell = n'' - n'$
for $n', n'' \in C_j$.
It follows that the contribution of each non well separated
term is $O(\rho^{ds}/|\ln\rho|^{100r})$. Since the number of
with s clusters
terms is $O(N^s |\ln\rho|^{r-s}) \le O(\rho^{-sd} |\ln\rho|^{r})$
the contribution of not well separated tuples is
negligible completing the proof of the proposition $\square$
Next we prove Prop. 3. Given A take large B
and let $N = B\ln|\ln\rho|$, $\hat{\rho} = |\ln\rho|^{-A}$
If $k \ge N$ then
$$\mu(d(x, f^k x) < \rho) \le \mu(d(x, f^k x) \le \hat{\rho}) \overset{(\triangle)}{\le} C\mu((x,y)\in\mathcal{M} : d(x,y) \le \hat{\rho})$$
$$\le \bar{C}\hat{\rho}^2$$
by exponential mixing
Exercise 11 prove $(\triangle)$ by approximating $1_{d(x,y)\le\hat{\rho}}$
by a smooth function.

Next we consider $k \le N$. Suppose

$d(x, f^k x) \le \rho$. Then

$d(x, f^{kl} x) \le C \rho L^{lk}$ where $L = \max_x \|df(x)\|$.

Taking an appropriate l we get that

$\exists \hat{N} \in [N, 2N]$ such that

$$d(x, f^{\hat{N}} x) \le C \rho L^{\hat{N}} \le \sqrt{\rho}$$

Then the same argument as before shows that

$$\mu(x, f^k x) \le \rho \le \mu_{(K,A)}(x, f^{\hat{N}} x \le \sqrt{\rho})_2 \le C \hat{\rho}^2.$$

Next if x is not slowly recurrent then

$\forall j_0 \ \exists j \ge j_0 \ \exists k \le K_j$ such that

$$\mu(B(x, 2^{-j}) \cap f^{-k} B(x, 2^{-j})) \ge \frac{1}{j^A} \quad (*)$$

Call the set of x satisfying $(*)$ $H_{j,k}$.

By the foregoing discussion

$\mu(H_{j,k}) \le \frac{1}{j^{2A}}$ and so Borel-Cantelli lemma

shows that a.e. point is (K,A) slowly recurrent.

Since (K,A) are arbitrary the proposition follows. $\square$

The proof of Thm 1 uses a standard argument common to most proofs of Poisson limit theorems in dynamics.

Namely the problem is reduced to studying the probability that the orbit of x

visits $B(x_0, \rho)$ at times $n_1, n_2 \dots n_r$. If ~~all~~ n_i's are far from each other (long returns) then the required estimate $\mu(B(x,\rho))^r$ follows by mixing.

The case where there are short returns ($\exists\, i, j$ with $n_i - n_j$ small) requires a separate geometric argument.

We also note that the clustering argument used in the proof of ~~..~~ Thm 1 can be used to obtain the following result.

Let $S_N(x) = \sum_{n=0}^{N-1} A(f^n x)$, where $A \in C^L(M)$

<u>Thm 4</u> (Björklund–Gorodnik) If f is multiply exponentially mixing then

$\frac{S_N - N\mu(A)}{\sqrt{N}}$ converges to a normal distribution.

We refer to the paper of Björklund–Gorodnik for detailed proof ~~....~~ which is valid not only for $\mathbb{Z}$ actions but for other groups of subexponential growth.