Random walks in random environment.

In the first part of the course we described the basic properties of Poisson processes, their relation with stable laws and also gave an example of proving convergence of number of visits to small balls to the Poisson distribution.

Next we discuss applications of the Poisson processes. We start with a classical model called one-dimensional random walk in random environment.

Fix $\varepsilon_0 > 0$ and let $\{p_n\}_{n\in\mathbb{Z}}$ be a sequence of iid random variables taking value in $[\varepsilon_0, 1-\varepsilon_0]$. Denote $q_n = 1-p_n \in [\varepsilon_0, 1-\varepsilon_0]$. Once $\{p_n\}$ is fixed we consider a Markov chain $\{X_t\}_{t\in\mathbb{N}}$ such that

$$P(X_{t+1} = n+1 \mid X_t = n) = p_n$$

$$P(X_{t+1} = n-1 \mid X_t = n) = q_n.$$

X_t is called random walk in random environment (RWRE) on $\mathbb{Z}$. RWRE on $\mathbb{Z}^d$, $d \geq 2$ is defined similarly. However RWRE in higher dimensions is only partially understood, while $d=1$ can be analyzed completely. Remarkably while it is expected that for $d \geq 2$ the Central Limit Theorem holds (possibly with anomalous scaling for $d=2$) the behaviour for $d=1$ is more complicated.

Prop 1 (Solanon 1975)

If $E \ln p_n > E \ln q_n$ then $X_t \to +\infty$ a.s;
if $E \ln p_n < E \ln q_n$ then $X_t \to -\infty$ a.s;
if $E \ln p_n = E \ln q_n$ then X_t is recurrent in the sense that it visits every site infinitely many times with prob. 1.

To prove the proposition denote $\alpha_n = q_n / p_n$.

Lemma 2 $\quad P_1(X_t = 0) = 1 \iff \sum_n \prod_{k=1}^{n} \alpha_k = \infty$

$$P_{-1}(X_t = 0) = 1 \iff \sum_n \prod_{k=1}^{n} \frac{1}{\alpha_{-k}} = \infty$$

$\triangleright$ of prop 1: By Law of Large Numbers

$$\frac{\ln \prod_{k=0}^{n} \alpha_k}{n} \to E[\ln q_0 - \ln p_0] = -a$$

Thus if a is positive then $\prod_{k=0}^{n} \alpha_k \le C_\varepsilon e^{-(a-\varepsilon)n}$ so the walker visits 0 only finitely many time eventually moving to $[1, \infty)$. On the other hand if $a \le 0$ then the common term of the corresponding series does not tend to 0, so the walk returns to 0 after each visit to the positive semi axis. A similar discussion of visits to $[-\infty, -1)$ completes the proof. $\square$

Pf of Lem. 2 We look for a sequence m_n such that m_{X_t} is a martingale. This means that

$m_n = p_n m_{n+1} + q_n m_{n-1}$. Denote $\Delta_n = m_{n+1} - m_n$.

Then $p_n \Delta_n = q_n \Delta_{n-1}$. Letting $\Delta_0 = 1$, $m_0 = 0$ we get $\Delta_n = \prod_{k=1}^{n} \alpha_k$. It follows that

$\sum_n \prod_k \alpha_k = \infty \iff m_n \to \infty$ as $n \to +\infty$.

By Optional Stopping theorem for martingales

$P_1(X$ visits n before returning to $0) = \frac{1}{m_n}$.

Now the result follows easily. $\square$

Exercise 1 Complete the proof of Lemma 2.

Thus the proof of Lemma 2 is quite straightforward. In fact the main contribution of Solomon was to show what X_t can go to ∞ with zero speed, the behavior which is impossible for usual random walk. In fact a more detailed description of the walk behaviour in the transient case was obtained by Kesten, Kozlov and Spitzer. To fix the notation consider the case $E \ln q_0 < E \ln p_0$.

Let s denote the positive solution of $E\left(\frac{q_0}{p_0}\right)^s = 1$ if it exists and let $s = +\infty$ otherwise

~~Exercise show that~~

Exercise 2 Denote $\eta(s) = E\left(\frac{q^s}{p^s}\right)$.
Show that $\eta(s)=1$ has at most 1 positive solution and there are no solutions iff $p_n > q_n$ with probability 1.

Hint Show that η is convex, $\eta(0)=1$, $\eta'(0)<0$ and $\eta(s)\to+\infty$ as $s\to\infty$ unless $p_n > q_n$ with probability 1.

Since we assume $E\ln q_0 < E\ln p_0$, $X_t \to \infty$ so X_t looks monotone at large scales, so one can understand the behaviour of X_t by considering $T_N = \min\{t : X_t = N\}$.

Thm 3 [KKS75] Suppose $\ln \alpha_n$ is non-arithmetic $\forall a,h$ $p(\ln\alpha_n \in a+h\mathbb{Z})<1$.

(a) If $s<1$ then $\frac{T_N}{N^{1/s}} \Rightarrow \mathcal{X}_s$ – positive stable law of index s.

(b) If $s\in(1,2]$ $\exists V$:
$\frac{T_N - \frac{N}{V}}{N^{1/s}} \Rightarrow \mathcal{X}_s$ – stable law of index s

(c) If $s>2$ then $\exists V$: $\frac{T_N - \frac{N}{V}}{\sqrt{N}}$ converge to normal distribution

(d) If $s=1$ then $\exists a_N$ s.t.
$\frac{T_N - a_N}{N}$ converges to stable law of index 1
Moreover $a_N \sim \mathrm{Const}\, N\ln N$

(e) If $s=2$ then $\exists v$ $\frac{T_N - \frac{N}{v}}{\sqrt{N \ln N}}$ converges to a normal random variable.

Remark Using standard tools one can convert the properties of T_N to properties of X_t, using a heuristic

$$P(X_t \le N) \approx P(T_N \ge t) \qquad (P)$$

In particular for $s>2$, X_t satisfies the Central Limit Theorem, for $s \in (1,2)$, X_t converges to stable distribution while for $s<1$ the limit distribution is called Mittag-Leffler and it defined using (P) namely

$$P(\mathcal{X} \le z) = P(\mathcal{Y} > z^{-1/s})$$ where $\mathcal{Y}$ is strictly positive stable random variable of index s.

For completeness let us mention the limiting behaviour in the symmetric case.

<u>Thm 4</u> (Sinai 1982) If $E \ln p_0 = E \ln q_0$ then $\frac{X_t}{(\ln t)^2}$ converges to a limit distribution called Kesten - Sinai distribution.

Our goal in this course is to describe the main steps in the proof of Theorem 3 (a) and then comment briefly on other parts.

The proof requires a number of background fact from rare event theory which are of independent interest.

Let $S_n = \sum_{k=1}^{n} Z_k$ where Z_k are iid compactly supported random variables with $EZ_k < 0$.

Define $\rho(\xi) = \ln E(e^{\xi Z})$ and let $I(a)$ be Legendre transform $I(a) = \xi_a a - \rho(\xi)$ where $\rho'(\xi_a) = a$.

Suppose the law of Z_K is non arithmetic.

<u>Lemma 5</u> $P(S_N > aN) = \frac{1}{\gamma(a)\sqrt{N}} e^{-NI(a)}(1+\bar{o}(1))$

where $\bar{o}(1)$ term is uniform for a in a compact interval not containing 0.

▷ The proof relies on change of measure technique due to Cramer. Let Z^{ξ} be the random variable with law $dP_{\xi} = \frac{e^{\xi Z}}{e^{\rho(\xi)}} dP$ where P is the law of Z.

<u>Exercise 3</u> Show that $\rho'(\xi) = E(Z^{\xi})$, $\rho''(\xi) = V(Z^{\xi})$.

We now take $\xi = \xi_a$, so that $E(Z^{\xi}) = a$. Then Take $y > 0$ and a small δ.

$$P(S_N - aN \in [y, y+\delta]) \approx \frac{E(e^{\xi S_N} 1_{S_N - aN \in [y, y+\delta]})}{e^{\xi aN + \xi y}}$$

$$= e^{-\xi y} e^{-NI(a)} P(S_N^{\xi} - aN \in [y, y+\delta])$$

where $S_N^{\xi} = \sum_{k=1}^{N} Z_k^{\xi}$ and Z_k^{ξ} are iid with law P^{ξ}

Next the local limit theorem for S_N^ξ tells us that

$$P(S_N^\xi - E S_N^\xi \in [y, y+\delta]) = \frac{1}{\sqrt{2\pi}} \delta$$

since Z_k^ξ have non arithmetic distribution

Dividing $[aN, \infty]$ into intervals
$I_j = [aN+j\delta, aN+(j+1)\delta]$ and summing over j
we obtain the result. $\square$

<u>Exercise 4</u> Complete the proof of Lemma 5.

Next we consider large deviations for the maximum of the walk. Let $M = \max_{n\geq 0} S_n$. It is well defined since $S_n \to -\infty$ as $n \to +\infty$. We want to find the asymptotics of $P(S_n > R)$ for large R.

We start with logarithmic asymptotics. To this end we want to see which n maximizes $P(S_n > R)$. We have

$$(P)\quad \ln P(S_n > R) \approx -n\, I\left(\frac{R}{n}\right) = -R \frac{I(a)}{a}, \quad a = \frac{R}{n}.$$

Thus we want to find $\operatorname{argmin} \frac{I(a)}{a}$. So we need a satisfying

$$\left(\frac{I(a)}{a}\right)' = \frac{a I'(a) - I(a)}{a^2}.$$

On the other hand by duality of Legendre transform if $I(a) = a\xi - \gamma(\xi)$, then $a I'(a) = I(a)$.

$I'(a) = \xi$ and so $a I'(a) - I(a) = \gamma(\xi) = 0$.

Plugging this to (P) we conclude that

$$\ln P(M > R) = -R\xi \text{ where } \gamma(\xi) = 0.$$

<u>Exercise 5</u> Show that $\exists C$ s.t. $P(M > R) \sim C e^{-\xi R}$

<u>Hint</u> Let n be the last time S_n is above R. Then

$$P(M > R) \simeq \sum_n \sum_j P(S_n \in [R + j\delta, R + (j+1)\delta]) q_j$$

where $q_j = P(\max_{n > 0} S_n < -(j+1)\delta)$

Show that the main contribution comes from $n = \frac{R}{\xi} + O(\sqrt{R})$ and the leading terms are of order $\frac{e^{-\xi R}}{\sqrt{R}}$ by Lemma 5.

With this background we are ready to discuss the distribution of the hitting time. The main reason why RWRE can be much slower that the simple random walk is the presence of traps. That is, even if $X_n \to +\infty$ there are still can be long intervals where drift pointing to the left. Consider for example the case where p_n is equally like to take values ε and $1-\varepsilon$? Then on the interval $[0, N]$ there segments of length $L_N \sim (\log 2) \ln N$ such that $p_n = \varepsilon$ on each point of that segment.

<u>Exercise 6</u> Show that $\forall K\ \exists \varepsilon$ s.t.

$P(X$ traverses the bad segment in time $\leq e^{KL_N}) \to 0$

<u>Hint</u> Let segment in question be $[a, b]$. Count how many time the walk visits a before reaching b the first time.

Thus by taking ε small one can achive that with probability close to 1 RWRE needs time more than N^K to reach N where K can be arbitrary large.

The above computation shows that RWRE can be subdiffusive for some values of the parameters but to understand the limit distributions one needs to make a precise definition of traps and show that they form a Poisson process. Let ρ_n be the expected number of visits to site n. To compute ρ_n let Z_n^+ be expected number of traverses of the edge $n \to n+1$ and Z_n^- be expect number of traverses of $n+1 \to n$. Then $Z_n^+ = p_n \rho_n$, $Z_n^- = q_{n+1} \rho_{n+1}$ and $Z_n^+ = Z_n^- + 1$ leading to

$$\rho_n = \frac{q_{n+1}}{p_n} \rho_{n+1} + \frac{1}{p_n} = \frac{1}{p_n} + \frac{q_{n+1}}{p_n}\left[\frac{1}{p_{n+1}} + \frac{q_{n+2}}{p_{n+1}} \rho_{n+2}\right]$$

$$= \dots = \frac{1}{p_n} \sum_{k=0}^{\infty} \prod_{j=n+1}^{n+k} \alpha_j = \frac{1}{p_n} Z_n$$

<u>Lemma 6</u> (Pareto asymptotics) If α_j is non arithmetic then

(a) $P(Z_n > R) = C_Z R^{-s}(1+\bar{\bar{o}}(1))$, as $R \to \infty$,

(b) $P(\rho_n > R) = C_\rho R^{-s}(1+\bar{\bar{o}}(1))$ as $R \to \infty$.

We note that the second claim follows from the first because $P(\rho_n > R) = \int P(Z_n > Rx)\, dP(p_n \leq x)$

To prove part (a) we write

$$Z_n = \sum_{k=1}^{\infty} e^{S_{n,n+k}} \quad \text{with } S_{m,n} = \sum_{k=m+1}^{n} \ln \alpha_k$$

$$\underset{=}{\text{Law}} \sum_{k=1}^{\infty} e^{S_k}$$

Exercise 7 Show that

$$P(S_m > M, S_{m+K} > M) \le C e^{-SM} \theta^k \text{ for some } \theta < 1$$

Hint Take a small ε and consider two cases:

(I) $S_m > M + e^{\varepsilon K}$ or (II) $S_m \le M + e^{\varepsilon K}$

Exercise 7 shows that the cheapest way to get $\sum_{k=1}^{\infty} e^{S_k} \ge R$ (*) is to have $M = \max_k S_k$ of order $\ln R$.

Indeed if (*) is achived then for some j we must have $\delta e^{\varepsilon j}$ terms with $S_K \in [\ln R - j, \ln R - j + 1)$ and the probability of such an event decays rapidly with j. This allows to approximate $P(\sum e^{S_k} \ge R)$ by $\sum_n P(A_{n,M,L} \ge R)$ for large M, L where

$$A_{n,M,L} = \sum_{k=n}^{n+L} e^{S_k} \underline{1}_{S_n \ge M,\, S_K \le M \text{ for } k \in [n, n+L]}$$

and the above probabilities could be controlled similarly to Exercise 5. □

We would like show that traps form a Poisson process. However ~~...~~ $Z_n \approx \alpha_{n+1} Z_{n+1}$ which shows that if Z_n is large then $Z_{n \pm 1}$ are also large. To account for this we will say that n is an edge of a trap if

$Z_n \geq \frac{N^{1/s}}{M}$ and $Z_{n-k} < \frac{N^{1/s}}{M}$ for $k \leq \varepsilon \ln N$

Here M is a large constant (which we let $\to \infty$ at the end) and ε is a small constant.

Now the key step in analysis of transient RWRE is

<u>Proposition 7</u> $\{n \leq N: n$ is (M, ε)-edge of a traps$\}$ form a Poisson process.

The proof can be executed using the approach of the first lecture. Namely for well separated tuples the events ~~$n_1, n_2, \dots n_p$~~ $\{n_j$ is an edge of a traps$\}$ are almost independent due to the following.

<u>Exercise 8</u> Let $Z = \sum_{k=1}^{\infty} e^{S_k}$ with S_k as above and $Z^{(n)} = \sum_{k=1}^{n} e^{S_k}$. Then $P(Z - Z^{(n)} > e^{-\varepsilon n}) \leq e^{-\varepsilon n}$

The contribution of non separated tuples is negligible due to definition if $|n_{j+1} - n_j| < \varepsilon \ln N$ and due to Exercise 7 if $|n_{j+1} - n_j| \geq \varepsilon \ln N$.

In fact we have the following extension of Prop 7.

Fix L and define the mass of the trap at n as $m_n = \sum_{k=1}^{L} p_{n+k}$.

<u>Proposition 7*</u> (a) As $N \to \infty$ $(n_j, m_{n_j}/N^{1/s})$ converges to a Poisson process with measure $dt\, \vartheta_{M,L}(m)$

(b) As $M \to \infty$, $L \to \infty$ ~~[illegible]~~ we have

$\vartheta_{M,L} \to \vartheta$ where $d\vartheta = c\,\frac{dm}{m^{s+1}}$

Indeed the proof of part (a) proceeds by conditioning on the value of Z_{n-L} and arguing as in the proof of Proposition 7 while part (b) follows because $m_n \overset{Law}{\approx} m_{n-L} \sum_{k=1}^{\infty} e^{Y_k}$ and the second term satisfies Pareto asymptotics.

Now we are ready to describe the limit distribution of $T_N/N^{1/s}$. Let ξ_n be the total time the walker spends at site n. The analysis of T_n involves the following steps

Step 1 $\forall \varepsilon\ \exists R$ s.t. $P(|T_N - \sum_{n=1}^{N} \xi_n| > R) \le \varepsilon$

Exercise 9 Prove this estimate

Step 2 Given a sequence n_j with $\rho_{n_j} \to \infty$ we have $\frac{\xi_{n_j}}{\rho_{n_j}} \overset{Law}{\to}$ Exponential random variables

This holds because ξ_{n_j} has geometric distribution with mean ρ_{n_j}.

Step 3 For n_j as on step 2 we have

$$P\left(\left|\frac{\xi_{n_j}}{\rho_{n_j}} - \frac{\xi_{n_j\pm1}}{\rho_{n_j\pm1}}\right| > \rho_{n_j}^{1/2+\kappa}\right) \le C_1 e^{-C_2 \rho_{n_j}^{2\kappa}}$$

This follows from moderate deviations for additive functionals of a 2 step Markov chain since we can consider our chain only at times T_{12}

it visits the sites n_j and $n_j \pm 1$.

Step 4 Let $\hat{\xi}_n$ be the number of visits to site n before the first visit to site $n + K \ln N$. Then for K large enough we have $P(\exists n \leq N: \hat{\xi}_n \neq \xi_n) \leq \frac{1}{N^{10}}$

Exercise 10 Prove this estimate using the arguments of Lemma 2.

Steps 1–4 show that the time spent in the traps is well approximated by $\sum_j m_j \mathcal{E}_j$ where m_j is the mass of traps and $\mathcal{E}_j$ are iid exponential random variables. Thus for $s<1$ the time spent in the traps divided by $N^{1/s}$ is well approximated by

$$\sum \left(\frac{m_j}{N^{1/s}}\right) \mathcal{E}_j \simeq \sum_j \tau_j \mathcal{E}_j$$

where τ_j is Poisson process with intensity C/τ^{s+1}.

By Exercise 7 the last sum has stable distribution with parameter s.

It remains to show that the time spent outside of traps is negligible. Namely let

$$\tilde{T}_{N,M} = \sum_{p_n < \frac{N^{1/s}}{M}} \xi_n.$$

Then $E(\tilde{T}_{N,M})$

$$= \int E\left(E_\omega(\tilde{T}_{N,M})\right) d(\text{Law of } p)$$

$$= E\left(\sum_{p_n < \frac{N^{1/s}}{M}} p_n\right).$$

By Pareto asymptotics the last sum $\boxed{B}$

can be bounded by

$$\text{Const } N \int_0^{N^{1/s}/M} m \frac{dm}{m^{1+s}} = \frac{C}{M^s}$$ which can be made small by taking M large.

If $1 < s < 2$ then we can not show that Expected time outside traps is negligible but we can show that its variance is small using similar argument.

Finally if $s > 2$ then we can divide the interval $[1, N]$ into segments of size N^θ for suitable θ separated by buffers of size $K \ln N$ for large K and show that if t_j is the time spent in j th large segment than t_j are asymptotically independent (since we can prohibit backtracking) random variables satisfying Feller Lindenberg condition leading to the Central Limit Theorem for T_N.

<u>Exercise 11</u> Let τ_N be the first visit to site n and $\tilde{\xi}_n$ be the number of crossings of edge $[N-n-1, N-n]$ before time τ_N. Show that $\tilde{\xi}_{n+1} = \sum_{k=1}^{\tilde{\xi}_n} Y_{k,n}$ where $Y_{k,n}$ are geometric random variables with parameter p_{N-n}. This allows to use the theory of branching processes to study RWRE.